Lecture 15

Wave propagation

15.1 Reflection

15.1.1 One-dimensional

As waves move through inhomogeneous media, which most of the world is, they reflect, bend, and attenuate. You have already seen waves reflecting off of fixed boundries in the previous problem set. Why do such reflections occur?

Mathematically, a fixed boundary is a boundary condition. And, specifically, a fixed boundary on a string is a condition that $u(l,t) = 0$ where l is the position of the boundary. So all we have to do to find the solution to the wave equation with a fixed boundary is force this mathematical condition at all times. In the previous problem set, we showed that a general solution to the wave equation can be written as

$$u(x,t) = f(x + vt) + g(x - vt),$$

Let's consider a case with an initial condition such that $u(x,0)$ is a single pulse centered at $x = 0$ which begins travelling in both directions (Figure 15.1.1). In this case, $g(x) = f(x)$, and the solution for a string with no boundries would be $u(x,t) = g(x - vt) + g(x + vt)$, where the $g(x)$ is the function of the initial pulse, and pulses would travel away to infinity. How do we, instead, satisfy the same initial conditions but also the fixed boundary condition?

Let's consider, for a second, an infinite string with an initial condition defined as

$$g'(x) = g(x) - g(2l - x).$$

The solution to the wave equation becomes

$$u(x,t) = g(x + vt) - g(2l - x + vt) + g(x - vt) - g(2l - x - vt)$$

Now we note that if $x = l$ this solution becomes

$$u(x,t) = g(l + vt) - g(l - vt) + g(l - vt) - g(l + vt) = 0$$

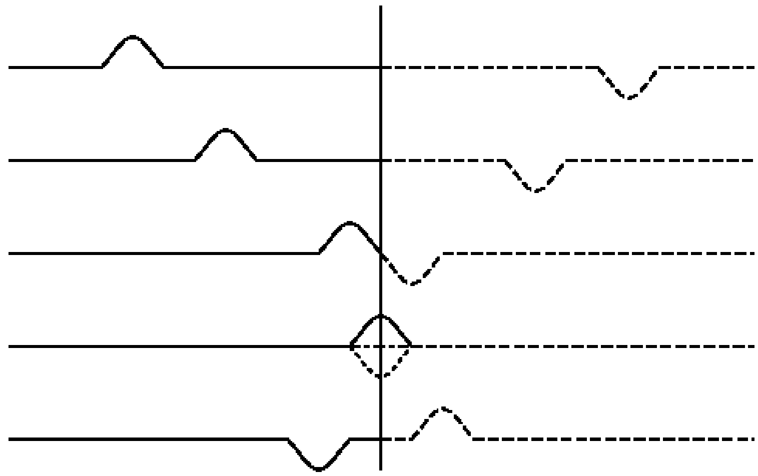

Figure 15.1: A pulse reflecting from a boundary is equivalent to the pulse meeting its mirror image at the boundary position.

so that solution is always zero at the boundary. Conceptually, this solution is simple to understand; the positively travelling pulse meets the negatively traveling pulse at point l and, because they are the opposite of each other, they cancel.

Now, mathematically, this same solution must also solve the the case where we force $u(l,t) = 0$. Here, the positively travelling wave "passes through" the boundary and disappears, and all we see left is the reflected neagtively travelling wave. Thus we see how a wave reflects at a fixed boundary. Is this really true, though? When we start a wave travelling off in one direction, is there really an imaginary wave started off in the opposite direction that is perfectly suited to cancel and reflect at the boundary? Don't be stupid, of course not. In real life, the accelerations and velocities work out so that waves are reflected in such a manner as to make this mathematical abstraction appear to be true. In the numeric solution to the wave equation done in the problem set reflections appeared without having to resort to weird imaginary waves. Nonetheless, it is sometimes mathematically and conceptually useful to think of the reflection as the superposition of an already existing negatively travelling wave. The math doesn't care what is *really* going on.

15.1.2 Plane wave

Most of the basic aspects of higher dimensional wave behavior can be determined by studying the effects of *plane waves.* A plane wave is the solution to the two-dimensional analog to the one-dimensional wave equation. The solution analogous to oscillating infinte string is the oscillating infinte plane, where now instead of wavenumber k, we have wavenumber $\mathbf{k}$, which is a vector because the waves are travelling in a particular direction, and the solution (or at least the positively-travelling half of the solution) is

$$u(\mathbf{x}, t) = u_0 \exp(i\mathbf{k} \cdot \mathbf{x} - i\mathbf{k} \cdot \mathbf{x}t)$$

where, here, we have allowed the possibility that v might also be a vector, meaning that the velocity is different in different directions.

What happens when a plane wave hits a boundary? Just like in the one-dimensional case, the plane wave reflects away from the boundary, but now, since we are in two dimensions, we must ask: in which direction does the plane wave reflect away from the boundary? We of course already know the answer: the angle of incidence must equal to the angle of reflection. We could prove this use the virtual wave method from above. Consider specifically a wall at $x = 0$ and a plane wave hitting it at angle I (measured from the normal to the wall). In this case $\mathbf{k} = k[\cos I, \sin I]$. The same boundry condition as usual applies at the wall: in all cases, $u(x = 0, t) = 0$. How can we make this so? Imagine superimposing another set of plane waves traveling from the other direction with opposite phase and with angle $R = -I$, such that $\mathbf{k} = k[-\cos I, \sin I]$. The two plane waves then have the form:

$$u_I = u_o \exp[ivk(x\cos I + y \sin I - xt - yt)$$

and

$$u_R = -u_o \exp[ivk(-x\cos I + y \sin I - xt - yt)$$

so that at the $x = 0$ boundry

$$u_I + u_R = u_o \exp[ivky(\sin I - t)] - u_o \exp[ivky(\sin I - t)] = 0.$$

The boundry condition is met at all times, so the solution to the left of the boundry must be the mathematically correct solution, showing us, as we already knew, that the angle of incidence and the angle of reflection must be the same

15.2 Refraction

We are now ready to consider the case of *refraction.* Refraction of a wavefront occurs when the wave crosses a boundary between regions where the wave velocities differ. That such a boudry crossing causes a bending of the wave is again easy to see by considering parallel strings hitting a barrier (Figure 15.2).

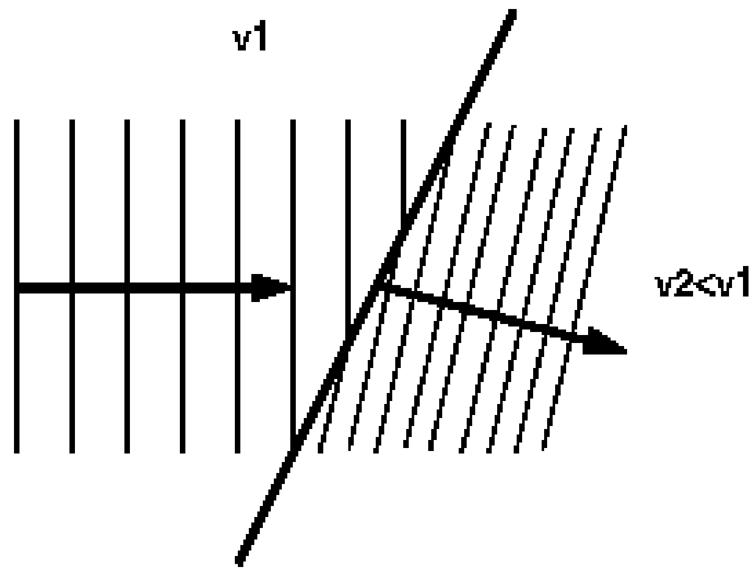

Figure 15.2: Refraction of parallel wavefronts as they pass from a region of higher to lower wave velocity.

The pulses that hit the boundary first are slowed so that when the later pulses hit the boundary they have travelled ahead of the first pulses and the wave direction, defined again as the normal to the line passes through all of the wave pulses, has bent.

What is the angle that the wave is bent by? Unfortunately, we cannot use our superposition method because we care physically what is going on on both sides of the barrier so we cannot add fake waves anywhere. Luckily, we can calculate this one directly pretty easily. Consider waves starting in a region with velocity v_1, hitting a boundary at angle θ_1 (defined as above), and passing into a region of velocity v_2. The geometry is illustrated in Figure 15.2. From the geometry we find that

$$d_1 = L \sin \theta_i$$

and

$$d_2 = L \sin \theta_r$$

But d_2 is just the distance that the wave travels in region 2 in the time it takes a wave in region 1 to travel d_1, or

$$d_2 = \frac{v_2}{v_1} d_1$$

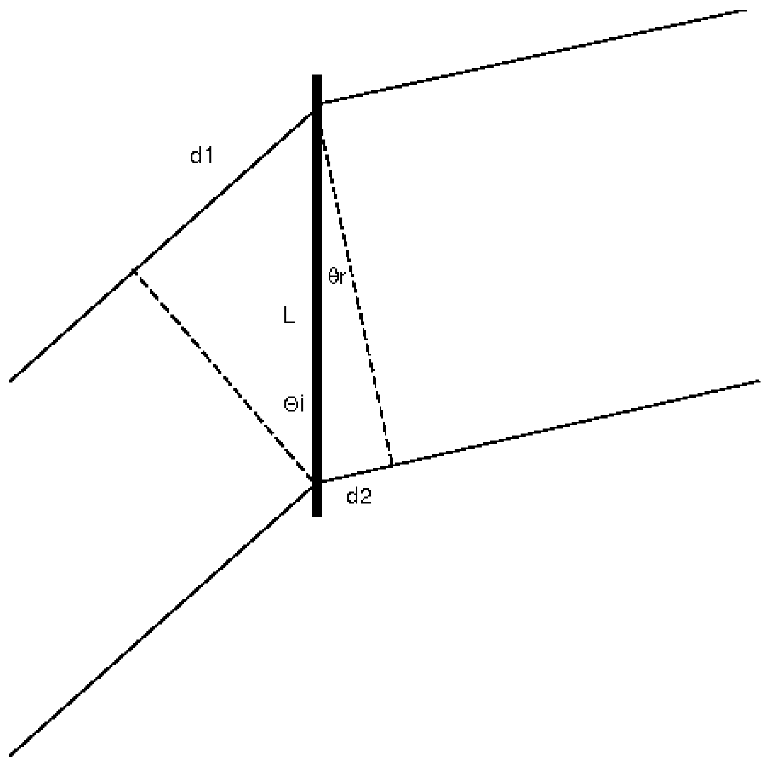

Figure 15.3: Refraction geometry.

This fact gives us the quick answer that

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{d_1}{d_2}$$

which becomes

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{v_1}{v_2}$$

which is *Snell's refraction law.*

In optics, you will often see Snell's law written as

$$\frac{\sin\theta_i}{\sin\theta_r} = n$$

where n is called the index of refraction of the material, defined as $n = v_1/v_2$ and $v_1 = c$, the speed of light in a vacuum, so n is really just the reciprocal of the speed of light in the medium divided by the speed of light in vacuum. Glass, for example, has an index of refraction of something like 1.5, meaning that light travels 50% slower in glass than in vacuum.

15.3 Transmission

So far, we have only determined the directions that waves go after reflection or refraction. But we have yet to determine how much of a wave is reflected or refracted at a boundary.

First, let's think of what a boundary is, anyway. What is a fixed wall? In one sense, a fixed wall is just a region where the velocity changes to zero. In this case we have found that the wave is reflected away from the boundry, and none is transmitted through. Think now of a conceptual boundrary between two strings that have the same wave velocity on them. The wave doesn't notice the difference between the two strings and the wave continues passing through the boundary, and nothing is reflected. What about something in between these two extremes? What is, for example, a wave encounters a region where the new wave velocity is very very low, but not zero, as would happen if a wave on light string suddenly came to a region where the string was very very heavy. Obviously, this case is almost like hitting a wall, so most of the wave would be reflected back, but a small portion would be transmitted forward with a much smaller wavelength and velocity (but with the same frequency). Similarly, if a wave hits a region with only very marginally heavier string, it will mostly transmit all the way through, but a tiny portion of the wave will be reflected.

Let's mathematically derive the amount of transmission and of reflection for the simple case of one string. A positively travelling pulse (for concreteness, this pulse will be defined as $g(x-vt)$) is launched from the (high velocity) side where $v=v_1$ and it hits a boundary at $x=0$ where suddently the mass of the string increases so the velocity of wave travel decreases to $v=v_2$ (Figure 15.3). We can solve this problem using the same method of superposition of pulses as we used for the pure reflection problem.

First, we will assume that the solution is one where after the pulse has hit the boundary, we have a reflected wave with relative height R travelling in the negative direction away from the boundary and a transmitted wave with relative height T travelling in the positive direction away from the boundary and with the wavelength decreased by the factor v_1/v_2 just as in the refraction case above. We will then come up with separate solutions for the two halves of the string and require them to match at the center.

For the left half solution, consider an infinite string with the initial pulse travelling on the left hand side, and an opposite pulse with height R travelling in the other direction. Mathematically, this situation becomes

$$u_1(x,t) = g(x - v_1 t) - Rg(-x - v_1 t).$$

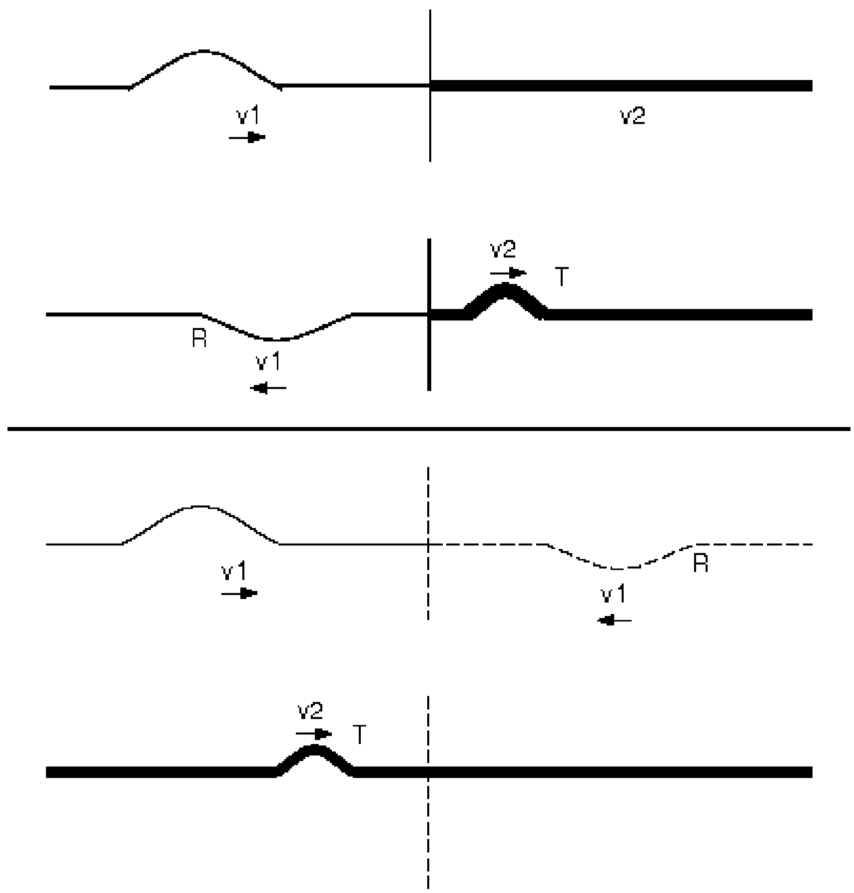

Figure 15.4: Transmission and reflection of waves can be considered as superposition of waves on two different infinite strings.

When these pulses pass, the left side will look just like it would if a pulse had reflected off the boundary with fractional height R. In fact, if you put out your hand and cover the right side of the string and watched the waves come and go, you would have no way of knowing whether there had been a reflection or if the two waves had initially started moving towards each other, just as in the pure reflection case that we considered above. This means that $u_1(x,t)$ *is* the solution for the left-side of the string whether there is a boundry or not. Of course, it doesn't tell us anything about what R actually is, though.

Now consider an infinite string with velocity v_2 and launch a pulse from the right side with initial condition

$$u_2(x,0) = g(\frac{v_1}{v_2}x)$$

which is the wavelength that the wave will have after it is transmitted through the boundry. Launch this wave in the positive direction, and the equation becomes

$$u_2(x,t) = Tg((\frac{v_1}{v_2}(x - v_2t)) = Tg(\frac{v_1}{v_2}x - v_1t).$$

Now cover the left side of this string and you will see, after some time, the wave pulse coming through where the boundary is. Again, there is no way to tell if the boundary is there and a reduced amplitude wave has been transmitted through or if instead there is no boundary. Again, then, we will assert that because this is true, u_2 must be the solution for the right side of the string, whether there is a boundary or not. And, again, this doesn't tell us what T acutally is.

Now we have these two separate solutions, and we can match them at the center to figure out R and T. First off, the values at the center must be equal:

$$u_1(0,t) = u_2(0,t)$$

which means that we must have

$$g(-v_1t) - Rg(-v_1t) = Tg(-v_1t)$$

which simply reduced to

$$R + T = 1$$

OK, we have at least shown that the sum of the amplitudes of the reflected and transmitted waves must be equal to the amplitude of the initial wave (and, incidentally, we have implicitly shown again that the reflected wave has a negative amplitude, since we explicitly made R refer to a negative amplitude wave). We could have probably figured this out without all of this effort, though. But we probably couldn't have done the next step.

Where do we get another boundary condition? Because we forced $u_1(0,t) = u_2(0,t)$ for a time, we also know that it must also be $\frac{d^n}{dt^n}u_1(0,t) = \frac{d^n}{dt^n}u_2(0,t)$ where $\frac{d^n}{dt^n}$ is an arbitrary time derivative. So we won't find any interesting boundary conditions there.

Thinking instead of spatial derivatives, we realize that the slope across the boundary must be a continuous function or else, since the acceleration is proportional to the 2nd derivative, there will be an infinite force at the non-continuous point. We can therefore force

$$\frac{d}{dx}u_1(0,t) = \frac{d}{dx}u_2(0,t)$$

or (with $\dot{g} = \frac{d}{dx}g$)

$$\dot{g}(-v_1 t) + R\dot{g}(-v_1 t) = \frac{v_1}{v_2} T\dot{g}(-v_1 t)$$

which shows that

$$T = \frac{v_2}{v_1}(1 + R)$$

to which we plug in above to reveal the final equations

$$\begin{aligned} T &= \frac{2v_2}{v_1 + v_2} \\ R &= \frac{v_1 - v_2}{v_1 + v_2} \end{aligned}$$

First, let's check if these make sense. If $v_1 = v_2$, no reflection should occur and the whole wave should be transmitted, and we see above that $T = 1$ and $R = 0$ as they should be. And if $v_2 = 0$ it should be like a wall and everything should reflect, which, since $T = 0$ and $R = 1$ it does.

For plane waves, the actual reflection and transmission amounts depend in detail on the actual angle that the boundry makes with the wave. As one can intuit, very steep angles lead to enhanced reflection, while straight-on impacts give you the best transmission of the wave.