Possible Observational Effects of Extra Forces in Extra Dimensional Models

額外次元模型內
由額外力所導致的可測量結果

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Abstract

Extra forces due to extra dimensional motions naturally arise in various kinds of extra dimensional models, where they could lead to deviations of particle trajectories compared to those without extra dimensions. In this thesis, we study the effect of extra forces on the classical particle trajectories in the five dimensional Friedmann universe and the Randall-Sundrum model. The classical trajectories observed in the ordinary dimensions are calculated explicitly in short time limits. In the $5D$ Friedmann universe with a non-static extra dimension, the deviations of the trajectories grow quadratically in time, and the compactness of the extra dimension does not show up because the extra dimension is conformally flat. In the Randall-Sundrum model where the extra dimension is inhomogeneous due to the warped geometry, beside the quadratic growth, the deviations also contain sinusoidal-like fluctuation, whose oscillation periods are approximately equal to the time of revolution around the compact extra dimension. As a classical test of the current models, we generate classical free wave packets in $5D$ space-times by randomizing the initial conditions of both ordinary and extra dimensional motions, and we compare their evolutions to those in the ordinary $4D$ space-time. It is shown that the widths of the wave packets in the Robertson-Walker and Randall-Sundrum models evolve in the same way as that in the ordinary $4D$ space-time in the limit of static universe, and correction terms arise if the size of the extra dimension changes with time.

We also study the extra force in the combined Kaluza-Klein-Randall-Sundrum model. It is shown that the extra force has a term that corresponds to the Lorentz force of electromagnetism, from which the charge-conjugation operation can be

defined as the parity reversal in the fifth dimension. It is also shown that, in the weak electromagnetic field limit, electric charges are equivalent to the fifth velocity, and charged and neutral particles can be represented by the same particle with the two different modes of fifth dimensional motions. The neutral particles would be observed to have masses suppressed by the warp factor, which generates a mass hierarchy between charged and neutral particles.

摘要

高維理論中，多存在由粒子在額外次元運動所致的額外力。這些額外力使粒子 軌跡偏離於一般四維理論中的軌跡。 在這份論文中，我們研究粒子在高維空間運動的經典軌道。我們採用兩個較熱 門的高維模型，包括廣義 Robertson-Walker 度規所形容的五維 Friedmann 宇宙，及翹曲額外次元的五維 Randall-Sundrum 空間。我們計算出一經典粒子 在這兩個高維模型中，投射在四維空間的軌道。在五維 Friedmann 宇宙中，軌道的偏移隨時間二次方增長，而額外次元的緊緻性並沒有展示出來。 在五維 Randall-Sundrum 空間中，軌道的偏移則除了二次方增長外，更有 波型振動項，振動週期大概相等於粒子行經一周緊緻化額外次元所需的時間。 為測試理論，我們把通常及額外的初始條件隨機化，得出一個在五維空間行進的 三維自由波包，並將其進化形態與四維空間的三維波包比較。我們指出波 包闊度的進化形式，在靜態宇宙極限下，跟四維的相同。當額外次元的尺度隨時間 改變，我們便得到其修正項。

我們亦討論於一個融合 Kaluza-Klein-Randall-Sundrum 模型中的額外力。我們證明這額外力當中，有對應Lorentz 電磁力的一個項，由此我們定義高維理論中的電荷共軛算子。 我們更指出，在弱電磁場極限中，帶電荷粒子和中性粒子是對應於 兩個不同模式的額外次元運動，而不帶電荷粒子的質量會被繞曲因 子所抑制。這樣自然地產生帶電荷粒子與不帶電荷粒子之間的質量 層級。藉此，我們嘗試，以微調一控制重力場和電磁場偶合強度的參數，了解中微子質量問題。

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Chapter 1

Prologue

1.1 Research Background

Extra dimensions were first employed in the attempt of unifying the gravitational and electromagnetic forces with a five dimensional space-time [1, 2, 3, 4]. In the so-called Kaluza-Klein theory of gravity, the vector potential of a classical electromagnetic field arises naturally if the curvature of the fifth dimension depends only on the usual $4D$ coordinates [1]. Famous examples where extra dimensions play a crucial role include the String Theory [5] and brane-world cosmology [6]. More recently it has been proposed that extra dimensions can be used to solve the Hierarchy Problem. One of the well-known models that achieve this goal was given by Arkani-Hamed, Dimopoulos and Dvali (ADD) [7], where the Planck scale is reduced to TeV scale by volumetric suppression. The size of the extra dimensions can be as large as the millimeter scale. A more sophisticated model is given by Randall and Sundrum (RS-I) [8], where the hierarchy between the Planck scale and the Standard Model-matter scale is generated by the warp factor of the extra dimension whose size remains at the Planck length. The RS-I model can be viewed as the low energy limit of Horova-Witten heteorotic string theory after the Calabi-Yau compactification [8]. In the original models of ADD and RS-I (brane-world), only bulk gravitons were allowed, but very soon later bulk gauge fields and bulk fermion fields were also included (*e.g.* in models with flat universal extra dimensions), and a number of the massive Kaluza-Klein gravitons and new gauge and fermion field quanta were predicted [9], which will serve as new kinds of signatures in future experiments such as the Large Hadron Collider and Linear Collider [10] as well as in neutrino mass measurements [11].

There exists another extra dimensional theory known as the Space-Time-Matter (STM) or Induced-Matter theory [12]. In STM theory, every curved 4*D* space-time can be embedded in an empty 5*D* space-time through Campbell's theorem [12, 13]. In other words, the rest mass observed in our 4*D* space-time is only a manifestation of the geometry of hyperspaces in the empty 5*D* space-time. It is interesting to notice that this could be an alternative to the Higgs mechanism [14]. STM theory suggests that a test particle (whose rest mass and energy does not modify the underlying geometry) freely moving in 5*D* space-time experiences an extra force related to its motion along the extra dimension if it is percieved by a 4*D* observer living on the brane [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. The extra force in STM theory was first defined in Ref. [15] and was later modifed in Ref. [22] so that it is consistent with 4*D* observations. Due to the extra force, the 4*D* observer concludes that the rest mass of the test particle must be varying with the 4*D* proper time. This implies that the fifth dimension is intimately related to the observed rest mass. It was shown that one can always define for a particle the 4*D* induced mass as the product of its 5*D* mass and its momentum in the fifth dimension. In particular, if the particle is 5*D* massless, then the 4*D* induced mass is proportional to the fifth momentum [22]. Furthermore, the weak equivalence principle for a test particle can only be valid in the limit of zero 4*D* induced mass [26].

The extra forces in the brane models have also been discussed by several authors [18, 21, 23, 22]. Youm [18] found the general form of the equations of motion of a bulk particle. He pointed out that a mechanism was needed to confine a particle to the TeV brane. Seahra [21] established such a mechanism by using the generalized centripetal force and asserted that a 4*D* observer can only tell whether the space-time is 5*D* by examining the anomalous torque of a gyroscope, an idea very similar to the Foucault pendulum in a non-inertia frame. The continuity of the extra forces across singular branes was shown by Ponce de Leon [22].

A very simple way to understand the existence of the extra forces is that the total momentum and the total energy are conserved in the higher dimensional space-time. In other words, individual components of the four-momentum, especially those observed in our ordinary space-time, may not be conserved due to the energy-momentum tranfers between the ordinary and extra dimensions. This may lead to observational signatures distinct from those in 4D theories. For example, the first order perihelion shifts of a test particle around a spherical massive object would contain, beside the usual Schwardchild term, a extra term related to the curvature of the extra dimension [27, 12].

Another way of understanding the existence of the extra forces is that in 5D space-time, the following identity holds: $u_A F^A = 0$. Hereafter in this paper, five dimensional quantities will be denoted by captial Latin indices $A, B, C, \ldots$ running over $0, 1, 2, 3, 4$. For example, a five-vector x^A will be used to denote the five-tuple $x^A = (t, x, y, z, \phi)$, where $x^4 \equiv \phi$ is the extra coordinate. Extracting the extra terms, one gets $u_\alpha F^\alpha = -u_4 F^4$, where four dimensional quantities are denoted by lower case greek indices $\alpha, \beta, \gamma, \ldots$ running over $0, 1, 2, 3$. The right hand side of this equation is zero in 4D space-time, and a non-zero value of which could lead to a violation of 4D energy-momentum conservation. Based on this observation, Wesson [28] derives a Heisenberg uncertainty relation of a massless 5D particle in position and momentum measurements by employing the canonical gauge [29] and identifying the fifth coordinate as the reciprocal of the induced 4D mass observed in the laboratory, which is equivalent to the Compton wavelength. He conjectures that a particle is virtual if its Compton wavelength is smaller than the length scale of the fifth dimension and it is real otherwise. Wesson later [30] takes another route by transforming Milne's metric into a "metric wave", which is a metric whose coefficients in the 4D parts are complex waves, and he concludes that for a 5D massless particle, de Broglie's relations can be recovered if the fifth coordinate is identified as the induced $4D$ mass of the particle. These two work are actually not the very first attempts to formulate the $4D$ Heisenberg uncertainty principle by using a $5D$ deterministic theory. It happens that Klein was the first one who considered the possibility of interpreting the Scrödinger equation as a five-dimensional classical wave equation, and thus relating Planck constant to the size of the extra dimension [2]. (See also the comments in the last chapter of Reference [12].) However, the fifth dimension employed in the STM calculations becomes an internal degree of freedom, as an induced-mass dimension rather than a spatial dimension. The identification of the fifth dimension as the reciprocal of the induced mass and the use of an imaginary metric wave can be viewed as the "quantizaton rules" in STM theory.

1.2 Statement of the Problem

In the previous work [15, 12, 31, 32, 21, 23, 22], the general terms of the extra forces in different five dimensional models are obtained through dimensional reduction of the geodesic equations. These extra forces in general depend on the particle kinetics, but the geodesic equations are usually left unsolved, and only the qualitative behaviors (e.g. their linear dependence on the four-velocities) of the extra forces are discussed. Although the general form of the extra forces appears very similar in different classes (e.g. flat and warped) of $5D$ metrics, their evolution might be very different because particle kinematics are different. Our aim in this thesis is to solve for the $5D$ classical trajectories of a free massive bulk particle in flat and warped space-times.

We will, as other author did [31, 32, 21, 22], take the view of a bulk observer who is able to see the bulk particle regardless of its position in the extra dimension. As pointed out by Youm [31], this assumption is in contrast to the brane models, where the $4D$ observers should be confined to the visible brane. However, it is this assumption that leads to interesting observational results.

Intuitively, a free particle running through a compactified extra dimension may show in the ordinary dimensions (through momentum tranfers discussed above) a periodic motion with a period that equals the time for completing a cycle in the compactified extra dimension. Since we expect that the extra dimension is too small to be observable, the period of motion may be extremely short, and the periodic motion percieved in the laboratory may appears as some sort of fast fluctuations that can only be characterized by a probability distribution. In addition, the initial conditions of the extra dynamical variables are uncontrollable in the laboratory and should be randomized to give the spectra of the observed 4*D* trajectories. In the line of these thoughts, we employ some common metrics and look for the time evolution of such trajectories.

1.3 Organization of the Thesis

In Chapter 2, we review Einstein's General Relativity of curved space and time, based on which we review how Kaluza and Klein develop their five dimensional model to explain electromagnetism as a purely geometrical effect. Then we survey several extra dimensional models, including the modern version of the Kaluza-Klein theory called the Space-Time-Matter theory, the Arkani-Hamed-Dimopoulos-Dvali scenario and the Randall-Sundrum scenario. In Chapter 3, we develop the necessary tools for the discussions of extra forces. We first review the definition of force in four dimensions. Then we are able to identify what we will call extra forces from the equations of motion in various extra dimensional models.

In Chapter 6, we study the trajectories in a more general version of Arkani-Hamed-Dimopoulos-Dvali world, namely the five-dimensional Friedmann-Robertson-Walker universe. In Chapter 6, we study the trajectories in the Randall-Sundrum five-dimensional space-time. In both models, we carry out the randomization as discussed above. We compare the 5*D* wave packets in these two models with those in four dimensional theories. The extra forces in a combined metric of Kaluza-Klein and Randall-Sundrum ones will be investigated in Chapter 7.

Chapter 2

Review of Various Extra Dimensional Models

2.1 General Relativity

After the publication of Special Relativity, Albert Einstein soon realized that Newton's law of gravitation is not Lorentz invariant and thus incompatible with Special Relativity. This incompatibility is due to the notion of absolute accelerations in Newtonian mechanics. In order to overcome it, Einstein developed the equivalence principle, asserting that gravitational acceleration is also a frame-dependent concept, and, therefore, it must also be treated in the framework of relativity. Most importantly, all law of physics must remain valid in all accelerating frames of reference. But this is not the end of the story. In the space-time around the Earth, for instance, the gravitational strengths are different at different radial distance from the centre of the Earth, implying that there are infinitely many different accelerating frames filling the whole space-time. By the equivalence principle, the motion of an object depends *not* on its own mass but solely on the gravitational strength at its position. This implies that it is the structure of the space-time, or the space-time geometry, that determines the motion of a particle. The presence of a massive object thus modifies the geometry of the space-time around it. In this way, gravitation becomes a field that fills up the whole space-time. Consequently, action-at-a-distance in Newtonian theory of gravitation does not exist in the new theory. Furthermore, Lorentz invaiance in Special Relativity is replaced by general covariance in General Relativity, just as Galilean invariance in Newtonian mechanics is replaced by Lorentz invariance in Special Relativity.

Curved Space-time

Riemannian geometry is used in General Relativity as a geometric theory of gravitation. In the space-time, the distance between two points $x^\mu = (t,x,y,z)$ and $x^\mu + dx^\mu = (t+dt, x+dx, y+dy, z+dz)$, where $\mu = 0,1,2,3$, is given by the metric equation:

$$ds^2 = \sum_{\mu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

where $g_{\mu\nu}$ is called the metric tensor. In this thesis, we will adopt the convention that lower Greek indices $\alpha, \beta, \gamma, \ldots$ run over $0,1,2,3$. The summation of the above equation, and other equations in this thesis, can be written in a more compact way by adopting Eintein's summation convention: whenever the same summation index exists as a lower index as well as an upper index, the summation over all possible values ($\mu = 0,1,2,3$ in this case) is implied. Thus Eq. (2.1) can be rewritten as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.1'}$$

The inverse of the metric tensor $g^{\mu\nu}$ is defined by $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$, where the Kronecker delta δ^μ_λ is given by

$$\delta^\mu_\lambda = \begin{cases} 0, & \text{if } \mu \neq \lambda; \\ 1, & \text{otherwise.} \end{cases} . \tag{2.2}$$

In Special Relativity, only Minkowski (flat) space-time is relevant. The corresponding metric tensor is given by the Minkowski metric:

$$\eta_{\mu\nu} = \text{diag}\{1, -1, -1, -1\}. \tag{2.3}$$

In this thesis, we shall adopt the "mostly negative" convention, such that the time dimensions have positive coefficients while spatial dimensions have negative coefficients.

Einstein's Equation and Particle Motions

According to General Relativity, the curvature of the space-time should be determined by the stress-energy source in the underlying space-time. Indeed, the metric tensor $g_{\mu\nu}$ is governed by Einstein's equation:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.4}$$

where $G_{\mu\nu}$ is the Eintein's tensor made out of $g_{\mu\nu}$, $T_{\mu\nu}$ is the stress-energy tensor, which is defined by

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}, \tag{2.5}$$

where ρ is the energy densities, p is the pressure in the spece-time, and u^{μ} is the four-velocity of the source [33]. The raising and lowering indices are given by the metric itself:

$$A_{\mu} = g_{\mu\nu}A^{\nu}, \quad A^{\mu} = g^{\mu\nu}A_{\nu}. \tag{2.6}$$

It is remarked that the solution to Einstein's equation is not unique. A simple example is that both $\delta_{\mu\nu}$ and $\eta_{\mu\nu}$ satisfy Eintein's equation in the empty space, where $T_{\mu\nu} = 0$. To fix the solution, one needs to impose *a priori* boundary conditions. In this simple example, we need to know beforehand that the underlying space-time is Minkowskian rather than Euclidean.

A particle moving in a gravitational field is now pictured as a particle moving *freely* in a curved space-time. In other word, the momentum of the particle should be constant in the curved space-time. In Minkowskian space-time, this can be stated as

$$dp^{\mu} = 0. \tag{2.7}$$

To transit from Special Relativity to General Relativity, we simply replace the derivative by covariant derivative:

$$dp^{\mu} + \Gamma^{\mu}_{\alpha\beta} p^{\alpha} dx^{\beta} = 0, \tag{2.8}$$

or in terms of the four vector x^{μ} and the derivatives in the 4D proper time s, it is more commonly expressed as

$$\frac{d^2 x^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^{\beta}}{ds} \frac{dx^{\beta}}{ds} = 0, \tag{2.9}$$

This implies that the resultant trajectory is exactly the geodesics in the curved spacetime. Eq. (2.9) will be the central equation in the rest of the thesis.

2.2 Kaluza-Klein Theory

One of the most intriguing ideas in modern physics is the unification of the four known interactions, *i.e.* the gravitational interaction, the weak interaction, the electromagnetic interaction and the strong interaction. The latter three have been unified successfully over the past few decades, and the resulting theory is called the grand unified field theory (GUT) accompanied by the standard model. GUT is, however, by no means complete. Gravitational effects have been completely ignored (even though the energy densities of the quanta would gravitate); otherwise the theory would be highly nonlinear, which is difficult to quantize and is not renormalizable.

The quest for a unified field theory could be dated back to 1873, when James Clerk Maxwell published his famous equations relating electricity and magnetism, revealing a deep symmetry among the two apparently independent physical fields, namely that a moving charge generates a magnetic field while a moving magnetic dipole generates an electric field. However, since all reference frames moving with constant velocities are equivalent, electromagnetism was finally unified when Albert Einstein developed his special relativity. In this picture, electricity and magnetism are just manifestations of one single rank-two tensor $F^{\mu\nu}$. Einstein also unified the concepts of time and the three spatial coordinates into a 4-vector $x^{\mu} = (t, x, y, z)$.

As is well known, Einstein spent half of his life in searching for a field theory which could unify gravitation and electromagnetism. The first seemingly successful unification of these two fields was first suggested by Kaluza [1] and then refined mathematically by Klein [2].

Kaluza's Anatz

The theory of Kaluza and Klein starts with a five dimensional space. Four of these five dimensions are the three ordinary spatial plus one time dimensions. The remaining one is an extra dimension, whose nature is under exploration. The line element of this $5D$ space is

$$d\hat{s}^2 = \hat{g}_{AB} dx^A dx^B \,, \quad (A, B = 0, 1, 2, 3, 4) \tag{2.10}$$

where the summation over all indices is implied when the same index label appears as both superscript and subscript. x^0 refers to the time coordinate, x^1, x^2 and x^3 the usual extended spatial dimensions and x^4 the extra dimension. We will adopt the convention that five dimensional quantities are denoted by hats and capital Latin indices $A, B, C, \ldots$ run over $0, 1, 2, 3, 4$. Kaluza asserts that if one takes the "cylinder condition", i.e. the metric is independent of x^4:

$$\hat{g}_{AB,4} = 0, \tag{2.11}$$

and $\hat{g}_{44} = 1$, then the Christoffel symbols of the first kind has a member

$$[4\mu, \beta] = \frac{1}{2}\left(\hat{g}_{4\alpha,\beta} - \hat{g}_{4\beta,\alpha}\right). \quad (\alpha, \beta = 0, 1, 2, 3) \tag{2.12}$$

He conjectures that if the metric component $\hat{g}_{4,\alpha}$ is proportional to the vector potential A_α, then Eq. (2.12) actually relates the geometry to the electromagnetic field tensor

$$F_{\mu\nu} = 2\,[4\mu, \nu]\,. \tag{2.13}$$

Klein [2] then extends Kaluza's idea with a more mathematical ground. One can rewrite the metric (2.10) into the equivalent form:

$$d\hat{s}^2 = \left(\hat{g}_{\alpha\beta} - \frac{\hat{g_{4\alpha}}\hat{g}_{4\beta}}{\hat{g}_{44}}\right) dx^\alpha dx^\beta + \hat{g}_{44}\left(dx^4 + \frac{\hat{g}_{4\alpha}}{\hat{g}_{44}} dx^\alpha\right)^2. \qquad (2.14)$$

This form has singled out the first term as the $4D$ line element on the right hand side. Indeed, the quantity

$$g_{AB} = \hat{g}_{AB} - \frac{\hat{g}_{4A}\hat{g}_{4B}}{\hat{g}_{44}} \qquad (2.15)$$

is orthogonal to the fifth coordinate, so that $g_{44} = g_{4A} = 0$ [12]. If $\hat{g}_{\mu\nu}$ are independent of x^4, then $ds^2 = g_{\alpha\beta} dx^{\alpha dx^\beta}$ is obviously the required $4D$ line element. Klein also notices that ds^2 so defined satisfies is invariant under the following coordinate transformations:

$$x^{\mu'} = \hat{f}^\mu\left(x^\mu\right), \quad x^{4'} = x^4 + \hat{f}^4\left(x^\mu\right), \qquad (2.16)$$

such that x^μ are actually the usual $4D$ space-time coordinates. According to Kaluza's ansatz, $\hat{g}_{4\alpha} = \kappa A_\alpha$ and $\hat{g}_{44} = -1$, where β is a constant. The metric is thus given by

$$\hat{g}_{AB} = \begin{pmatrix} g_{\mu\nu} - \kappa^2 A_\mu A_\nu & -\kappa A_\mu \\ -\kappa A_\nu & -1 \end{pmatrix}, \qquad (2.17)$$

where $g_{\mu\nu}$ satisfies the 3+1 Einstein equation, A_μ is the vector potential of an electromagnetic field. The Einstein-Hilbert action can be evaluated in the five dimensional vacuum space:

$$\hat{S} = -\frac{1}{16\pi\hat{G}}\int d^5x\sqrt{\hat{g}}\hat{R} = -\int d^4x\sqrt{g}\Big(\frac{1}{16\pi G}R + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Big), \qquad (2.18)$$

where $\hat{G}$ is the five dimensional gravitational constant, $G = \hat{G}/\int dx^4$ is its four dimensional counterpart and R ($\hat{R}$) is the 3+1 (4+1) Ricci tensor of $g_{\mu\nu}$ ($\hat{g}_{\mu\nu}$), and $\kappa = \sqrt{16\pi G}$ have been set in order to get the usual electromagnetic action. Eq. (2.18) is just the sum of the actions of gravity and electromagnetism, indicating that the two forces have seemingly been unified by using $\hat{g}_{AB}$. Notice that the $5D$ space-time is vacuum but contains two kinds of $4D$ matter. This is remarkable, and it has been the basic assumption in the Space-Time-Matter theory, which will be discussed in Section 2.3.

Physical Attributes of the Fifth Extra Dimension

However, the postulation of the fifth dimension was quite artificial. It appeared to be purely a mathematical manifestation which incorporated electromagnetism with gravity. The first attempt to assign physical attributes to the extra dimension was made by Einstein and Bergmann [34]. They reason that if the fifth dimension is extremely tiny compared to the other three ordinary spatial dimensions and is periodic (compactified), then the total number of dimensions are effectively reduced to (3+1)D by intuition.

Indeed, Klein was inspired by the periodicity of the fifth coordinate well-before Einstein and Bergmann's work. He tried to reproduce the Schrodinger equation by taking the geometrical optical limit of a classical five-dimensional wave equation for massless particle and assuming that the fifth coordinate is periodic [2]. Furthmore, the canonical 4-momentum under the metric (2.17) is

$$p_\mu = m\frac{dx^\mu}{ds} + \kappa A_\mu p_4. \tag{2.19}$$

Thus this is the canonical momentum for an electron provided that

$$p_4 = e/\kappa. \tag{2.20}$$

If the fifth coordinate has a period b, then, from quantum theory, $p_4 = Nh/b$, where N is the quantum number, and h is the Planck constant. It follows that, in Gaussain units,

$$b \sim \frac{\kappa h}{e} \equiv \frac{\kappa h}{ce} \approx 10^{-30}\text{cm}, \tag{2.21}$$

which says that the fifth dimension must be extremely tiny. (It happens that this order of magnitude also coincides with the Planck scale.) Charge now has dimension

$$(\text{Mass})^{\frac{1}{2}}(\text{Length})^{\frac{3}{2}}(\text{Time})^{-1}. \tag{2.22}$$

2.3 Space-Time-Matter Theory

In the Kaluza-Klein theory, the cylindrical condition is critical to reproduce the vector potentials as the non-diagonal metric components. It is also this condition that leads to the compactification of the extra dimension. A natural generalization of the Kaluza-Klein theory is to release the cylindrical condition, *i.e.* we allow the metric components to be dependent on the extra coordinate. The resulting theory is called the modern Kaluza-Klein theory, or better known as the Space-Time-Matter (STM) theory or Induced-Matter theory [12]. Beside the release of cylinder condition, STM theory is inspired by another fact from the Kaluza-Klein theory: The higher dimensional space-time can be empty, yet appearing non-empty in the lower dimensional subspace-time. This assertion is supported by Campbell's Theorem [13]. A complete survey of the recent developments in STM theory can be found in [12].

5D Vacuum and 4D Induced Matter

The basic ansatz of the Space-Time-Matter Theory is that the 5D Einstein's equation satisfies:

$$\hat{G}_{AB} = 0, \tag{2.23}$$

or, equivalently,

$$\hat{R}_{AB} = 0, \tag{2.23'}$$

where $\hat{g}_{AB}$ is the five dimensional metric, $\hat{G}_{AB} = \hat{R}_{AB} - \hat{R}\hat{g}_{AB}$ is the five dimensional Einstein Tensor, $\hat{R}_{AB}$ and $\hat{R} = \hat{g}_{AB}\hat{R}^{AB}$ the five dimensional Ricci tensor and scalar respectively. (2.23) and (2.23') can be derived from varying the five dimensional Einstein action

$$S = \frac{1}{16\pi\hat{G}} \int d^5x \hat{R}\sqrt{\hat{g}}, \tag{2.24}$$

where $\hat{G}$ is a five dimensional gravitational constant.

The ansatz (2.23) asserts that the higher dimensional space-time is empty. It implies that matter observed in four dimensions are manifestations of pure geometry in higher dimensions. To illustrate this, the five dimensional Einstein's equation can be decomposed into usual four dimensional quantities plus the extra ones as follows.

In general, the metric $\hat{g}_{AB}$ has 25 components. A fully covariant five dimensional theory has 5 coordinate degrees of freedom, which can be used to simplify the algebraic calculations. A natural choice is specified by $g_{4\alpha} = 0$, leaving one of the coordinate degrees of freedom over for further simplifications. Compared to the Kaluza-Klein theory, this choice of coordinate degree of freedom removes the explicit electromagnetic potentials. Therefore such a choice in some sense describes neutral matter. The metric components may depend on the fifth coordinate: $\hat{g}_{\alpha\beta} = \hat{g}_{\alpha\beta}(x^A)$, $\hat{g}_{44} = \hat{g}_{44}(x^A)$, and thus the restriction of the cylindrical conditions in the original Kaluza-Klein theory is released. Notice that $\hat{g}_{\alpha\beta}$ is identified to be the $4D$ metric components $g_{\alpha\beta}$.

The five-dimensional line element can be written as

$$d\hat{s} = \hat{g}_{AB} dx^A dx^B, \tag{2.25}$$

where

$$\hat{g}_{\alpha\beta} = \begin{pmatrix} g_{\alpha\beta}(x^A) & 0 \\ 0 & -\Phi^2(x^A) \end{pmatrix}. \tag{2.26}$$

We assume that the fifth dimension is space-like. In Space-Time-Matter theory, it can be time-like or space-like, both of which may lead to well-behaved classical solutions. The $5D$ Ricci tensor in terms of the $5D$ Christoffel symbols is given by

$$\hat{R}_{AB} = \partial_C \hat{\Gamma}^C_{AB} - \partial_B \hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB} \hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD} \hat{\Gamma}^D_{BC}. \tag{2.27}$$

Writing out the $\alpha\beta$-, $\alpha 4$- and 44-parts explicitly and collecting the terms into the

$4D$ Ricci tensor, one gets [1]

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{\nabla_\beta \partial_\alpha \Phi}{\Phi} - \frac{1}{2\Phi^2}\left(\frac{\partial_4\Phi \partial_4 g_{\alpha\beta}}{\Phi} - \partial_4^2 g_{\alpha\beta} + g^{\lambda\mu}\partial_4 g_{\alpha\lambda}\partial_4 g_{\beta\mu} - \frac{g^{\mu\nu}\partial_4 g_{\mu\nu}\partial_4 g_{\alpha\beta}}{2}\right); \tag{2.28}$$

$$\hat{R}_{44} = \Phi\Box\Phi - \frac{\partial_4 g^{\lambda\beta}\partial_4 g_{\lambda\beta}}{4} - \frac{g^{\lambda\beta}\partial_4^2 g_{\lambda\beta}}{2} - \frac{\partial_4\Phi}{2\Phi} g^{\lambda\beta}\partial_4 g_{\lambda\beta}; \tag{2.29}$$

$$\hat{R}_{\alpha 4} = \sqrt{g_{44}}\nabla_\beta\left(\frac{g^{\beta\sigma}\partial_4 g_{\sigma\alpha} - \delta^\beta_\alpha g^{\mu\nu}\partial_4 g_{\mu\nu}}{2\sqrt{g_{44}}}\right). \tag{2.30}$$

Here ∇ denotes covariant derivatives and $\Box$ is the $5D$ d'Alembertian. By the assertion $\hat{R}_{AB} = 0$, we find from Eq. (2.28) that

$$R_{\alpha\beta} = \frac{\nabla_\beta(\partial_\alpha\Phi)}{\Phi} + \frac{1}{2\Phi^2}\left\{\frac{(\partial_4\Phi)(\partial_4 g_{\alpha\beta})}{\Phi} - \partial_4^2 g_{\alpha\beta} + g^{\lambda\mu}(\partial_4 g_{\alpha\lambda})(\partial_4 g_{\beta\mu}) - \frac{g^{\mu\nu}(\partial_4 g_{\mu\nu})(\partial_4 g_{\alpha\beta})}{2}\right\}. \tag{2.31}$$

That is, the empty $5D$ space-time is apparently materialised in $4D$ space-time. The $5D$ equation $\hat{R}_{44} = 0$ gives a scalar wave equation for the scalar field Φ, and $\hat{R}_{4\alpha} = 0$ gives a conservation law in $5D$. The $4D$ Ricci curvature scalar $R = g^{\alpha\beta}R_{\alpha\beta}$ is given by

$$R = -\frac{1}{4\Phi^2}\left[(\partial_4 g^{\mu\nu})(\partial_4 g_{\mu\nu}) + (g^{\mu\nu}\partial_4 g_{\mu\nu})^2\right]. \tag{2.32}$$

With Eqs. (2.32) and (2.28), we can define an $4D$ energy-momentum tensor by $8\pi T_{\alpha\beta} = R_{\alpha\beta} - Rg_{\alpha\beta}/2 \equiv G_{\alpha\beta}$. This gives

$$8\pi T_{\alpha\beta} = \frac{\nabla_\beta(\partial_\alpha\Phi)}{\Phi} + \frac{1}{2\Phi^2}\left\{\frac{(\partial_4\Phi)(\partial_4 g_{\alpha\beta})}{\Phi} - \partial_4^2 g_{\alpha\beta} + g^{\lambda\mu}(\partial_4 g_{\alpha\lambda})(\partial_4 g_{\beta\mu}) - \frac{g^{\mu\nu}(\partial_4 g_{\mu\nu})(\partial_4 g_{\alpha\beta})}{2} + \frac{g_{\alpha\beta}}{4}\left[(\partial_4 g^{\mu\nu})(\partial_4 g_{\mu\nu}) + (g^{\mu\nu}\partial_4 g_{\mu\nu})^2\right]\right\}. \tag{2.33}$$

[1] The simplifications of these expressions involve the use of (i) $(\delta^\mu_\nu)_{,4} = 0$, or equivalently, $g^{\mu\beta}g^{\lambda\sigma}\partial_4 g^{\lambda\beta}\partial_4 g_{\mu\sigma} + (\partial_4 g^{\mu\nu})(\partial_4 g_{\mu\nu}) = 0$; and (ii) $(\hat{g}_{44}\hat{g}^{44})_{,\alpha} = 0$ or $(\partial_4\hat{g}^{44})(\partial_\alpha\hat{g}_{44}) - \partial_\alpha\hat{g}^{44})(\partial_4\hat{g}_{44}) = 0$.

Thus the 4D ordinary space-time may seem non-empty. This illustrates the idea that matter observed in four dimensions may be induced from the geometry in a higher dimensional space-time.

The Canonical Gauge

In the above, the gauge conditions $g_{4\alpha} = 0$ exhaust only four of the five coordinate degrees of freedom. The remaining coordinate degree of freedom can be used to fix

$$\Phi(x^A) \equiv 1. \tag{2.34}$$

By redefining the 4D metric components $g_{\alpha\beta} \to l^2 g_{\alpha\beta}/L^2$, where $l = x^4$ and L is a length scale of the extra dimension, we get the "canonical" metric

$$d\hat{s}^2 = \frac{l^2}{L^2} g_{\alpha\beta}(x^\gamma, l) dx^\alpha dx^\beta - dl^2. \tag{2.35}$$

The algebra under this gauge is greatly simplified.

The canonical metric can be applied to the field equations. For example, if the canonical metric is substituted into (2.33), and $g_{\alpha\beta}$ is assumed to be independent on $x^4 = l$, then we find that Einstein's equation becomes $G_{\alpha\beta} = 3g_{\alpha\beta}/L^2$, implying a cosmological constant $\Lambda = 3/L^2$.

2.4 Arkani-Hamed-Dimopoulos-Dvali Scenario

While extra dimensions were originally proposed to unify the forces of nature, they enter the modern theory of elementary particles for a completely different reason. When we try to unify the four fundamental forces of nature, we often face the very question, which is actually the main difficulty in the unification: why gravity is so much weaker than the other interactions? One of the theories employing extra dimensions to tackle this problem is proposed by Arkani-Hamed, Dimopoulos and Dvali [35, 36].

The Hierarchy Problem

Each theory of interaction has a typical energy scale, above which the quantum effect of that kind of interaction becomes important. The typical energy scale in quantum chromodynamics is 100 MeV, and the electroweak energy scale is of order 100 GeV. The current energy scale that is accessible in the laboratory is 10 TeV, and these two theories describing three different interactions are shown to be consistent with experimental results. In gravitational interaction, the energy scale can be estimated from the gravitational constant, the Planck constant and the speed of light

$$E_G = \sqrt{\frac{\hbar c^5}{G}} \sim 10^{19}\text{GeV}. \tag{2.36}$$

E_G is often called the Planck energy, which is an order of magnitude 10^{17} higher than the electroweak scale and is far away from the current technological accessible energy range. This is called the Hierarchy Problem because the gravitational scale differs too much from other interactions.

Enter Extra Dimensions

Arkani-Hamed, Dimopoulos and Dvali (ADD) proposed a solution to the Hierarchy Problem by employing the extra dimensions. Their central idea is to treat the electroweak energy scale as the fundamental scale for the gravitational interaction in the extra dimensional world. Then the effective coupling strength in 4D may be suppressed to the Planck scale by the volume of the compact extra dimensions.

Suppose that the spacetime consists of 1+3+n dimensions, where the $1+3$-dimensional subspace is the ordinary spacetime, and each of the other n flat dimensions are compact with a common size R. Since the properties of the standard matter (SM) fields have been tested rigorously in the four dimensional subspace, the authors assert that the SM fields should be confined on the 4D subspace (or brane); only gravitons are able to propagate through the extra dimensions. For $r < R$, the potential energy between two test masses m_1 and m_2 is given by the Gauss law and reads

$$V(r) \sim \frac{m_1 m_2}{M_G^{n+2}} \frac{1}{r^{n+1}}, \tag{2.37}$$

where M_G is the energy scale of gravitation in the $n+4$ spacetime, which is of electroweak scale by assumption. For $r > R$, the potential now reads

$$V(r) \sim \frac{m_1 m_2}{M_G^{n+2} R^n} \frac{1}{r}. \tag{2.38}$$

so that the usual inverse-r law is recovered. We see that the Planck scale is related to M_G via

$$M_{\rm Pl}^2 = M_G^{n+2} R^n. \tag{2.39}$$

If M_G is the same as the electroweak scale ($\sim$ TeV)

$$R = \left(\frac{M_{\rm Pl}^2}{M_G^{n+2}} \right)^{\frac{1}{n}} \sim 10^{\frac{30}{n}-17} \text{ cm}. \tag{2.40}$$

If $n = 1$, then $R \sim 10^{13}$ cm which is of solar system order. If $n = 2$, then $R \sim 1$ mm. Thus $n \geq 2$ implies the possibility of flat extra dimensions with submillimeter scales that solve the Hierarchy Problem.

Kaluza-Klein Decompositions

Since the extra dimension is also compactified in the ADD brane model, quantum mechanics suggests that a particle would have extra momenta which are multiples of the inverse sizes of the extra dimensions. (Klein also exploits this quantization of wave length to estimate the size of the extra dimension. See Section 2.2.) In quantum field theory, this is translated as that the particle would have mass eigenmodes that do not appear in 4*D* theories. We will show below how to get these new mass eigenmodes of a classical field in an ADD braneworld. These eigenmodes are often called the Kaluza-Klein modes. For simplicity, we would employ a 5*D* toy model, where the extra dimension is flat.

A metric appropriate for describing a 5*D* ADD braneworld is

$$d\hat{s}^2 = g_{AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu - b^2 d\phi^2, \tag{2.41}$$

where $A,B = 0,1,2,3,4$, $mu, \nu = 0,1,2,3$, b is the size of the extra dimension, and g_{AB} is the $4D$ metric with "mostly negative" trace. The range of ϕ is $[-\pi,\pi]$. The Klein-Gordon equation for a classical field Φ with mass m in ADD model is

$$\left(g^{\mu\nu}\partial_\mu\partial_\nu - \frac{1}{b^2}\partial_\phi^2 - m^2\right)\Phi = 0, \tag{2.42}$$

where g^{AB} is the inverse of g_{AB}, and Φ is a classical field. The Poincaré invariance of the $4D$ ordinary space-time implies that Φ can be expanded as a direct sum of the products of eigenfunctions in x^μ and ϕ. Furthermore, the compactness of the extra dimensions allows us to expand the eigenfunctions in ϕ into Fourier series:

$$\Phi(x^\mu, \phi) = \sum_n e^{in\phi}\Phi^{(n)}(x^\mu). \tag{2.43}$$

Then the effective Klein-Gordon equation for the n-th Kaluza-Klein mode is

$$\left(g^{\mu\nu}\partial_\mu\partial_\nu + \frac{n^2}{b^2} - m^2\right)\Phi = 0. \tag{2.44}$$

Therefore, in the effective theory, the higher dimensional field appears as an infinite tower of fields with masses $m_n^2 = m^2 + n^2/b^2$. We will see in Chapter 3 that when the $5D$ classical trajectories are examined, the $4D$ rest mass is also varying with the fifth momentum, which is analogous to the generation of mass eigenstates here.

Notice that the mass splitting in the current metric is uniform. This is a consequence of the flat extra dimension. If the extra dimension is curved, then the mass splitting would be much complicated. This is the case in the Randall-Sundrum model.

2.5 Randall-Sundrum Scenario

In ADD model, the extra dimensions are flat, and the gravitational coupling is suppressed by the volume of the extra dimensional subspace. In sizes of the extra dimension there can be as large as millimeter scale, which is significantly larger than the Planck scale. Randall and Sundrum [37, 38] later employ a non-factorizable, warped metric to solve the Hierarchy Problem. The length scale of the Randall-Sundrum scenario can be made to remain as the Planck scale.

The Setup

The original Randall-Sundrum model deals with an orbifolded fifth dimension ϕ assigned with values lying in the interval $[-\pi,\pi]$. The basic assumption is that all $(3+1)$ gauge fields are contained in the 3-branes located at $\phi=0$ and π, which are conventionally called "hidden" and "visible" branes respectively (Fig. 2.1). Such brane setting was first employed in the superstring theory to remove anomalies [39, 40]. The bulk has a 5*D* Planck scale M and a cosmological constant Λ and it is described by the classical action

$$S_{\text{gravity}} = \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{\hat{g}}\{-\Lambda + 2M^3\hat{R}\}, \tag{2.45}$$

where $\hat{g}$ is the determinant of the bulk metric $\hat{g}_{MN}(M,N=0,1,2,3,4)$ and $\hat{R}$ is the 5*D* Ricci Scalar derived from $\hat{g}_{MN}$. The branes have, apart from the $(3+1)$ fields' Lagrangians, "vaccum energy" as sources in the absence of particle excitations. The classical actions for the 3-branes are

$$\begin{aligned} S_{\text{vis}} &= \int d^4x \sqrt{g_{\text{vis}}}\{\mathscr{L}_{\text{vis}} - V_{\text{vis}}\}, \\ S_{\text{hid}} &= \int d^4x \sqrt{g_{\text{hid}}}\{\mathscr{L}_{\text{hid}} - V_{\text{hid}}\}, \end{aligned} \tag{2.46}$$

where $g_{\text{hid}} = \det\hat{g}_{MN}(\phi=0)$ and $g_{\text{vis}} = \det\hat{g}_{MN}(\phi=\pi)$ are the determinants of the induced metric on the respective branes. The Einstein equation is given by

$$\begin{aligned} &4M^3\sqrt{\hat{g}}(\hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}) - \Lambda\sqrt{\hat{g}}\hat{g}_{MN} \\ &= V_{\text{vis}}\sqrt{\hat{g}_{\text{vis}}}\hat{g}^{\text{vis}}_{\mu\nu}\delta^{\mu}_{M}\delta^{\nu}_{N}\delta(\phi-\pi) + V_{\text{hid}}\sqrt{\hat{g}_{\text{hid}}}\hat{g}^{\text{hid}}_{\mu\nu}\delta^{\mu}_{M}\delta^{\nu}_{N}\delta(\phi). \end{aligned} \tag{2.47}$$

Randall and Sundrum find the static, 4-dimensionally flat metric for the above action. They assume that the line element has the form

$$d\hat{s}^2 = e^{-2kb\omega(\phi)}\eta_{\mu\nu}dx^{\mu}dx^{\nu} - b^2 d\phi^2, \tag{2.48}$$

Figure 2.1: Pictorial description of the 2-brane world scenario in Randall-Sundrum model. The physical world visible to we human is located at $\phi = \pm\pi$ and a hidden brane is at $\phi = 0$.

where b is a "radius" of the fifth dimension and the constant k has a dimension $(\text{length})^{-1}$. The Einstein equations then read

$$\begin{aligned}\omega'^2 &= -\frac{\Lambda}{24M^3k^2},\\ \omega'' &= \frac{V_{\text{vis}}}{12M^3k}\delta(\phi-\pi)+\frac{V_{\text{hid}}}{12M^3k}\delta(\pi).\end{aligned} \tag{2.49}$$

The periodicity in ϕ requires that

$$V_{\text{hid}} = -V_{\text{vis}} = 24M^3k\,, \quad \Lambda = -24M^3k^2 \tag{2.50}$$

and

$$\omega(\phi) = |\phi|. \tag{2.51}$$

Physical Mass Scales

The four-dimensional effective theory can be examined by considering a tensor fluctuation

$$d\hat{s}^2 = e^{-2kb|\phi|}\overline{g}_{\mu\nu}(x^\gamma)dx^\mu dx^\nu - b^2 d\phi^2, \tag{2.52}$$

where $\overline{g}_{\mu\nu}$ is the metric describing the tensor fluctuation. In literature, it is usually written as a sum of the 5D Minkowski metric $\overline{\eta}_{\mu\nu}$ and a tensor fluctuation $\overline{h}_{\mu\nu}$: $\overline{g}_{\mu\nu}(x^\gamma) = \overline{\eta}_{\mu\nu}(x^\gamma)+h_{\mu\nu}(x^\gamma)$. By integrating ϕ in that action, we find

$$\begin{aligned}S_{\text{gravity}} &\supset 2\int d^4x \int_{-\pi}^{\pi} d\phi M^3 e^{-2kb|\phi|}\sqrt{\overline{g}}\overline{R}\\ &= 2\int d^4x \left[\frac{M^3}{k}\left(1-e^{-2kb|\phi|}\right)\right]\sqrt{\overline{g}}\overline{R},\end{aligned}$$

where $\overline{R}$ is the Ricci scalar made out of $\overline{g}$. Thus we can define a 4D Planck mass scale as

$$M_{\text{Pl}}^2 = \frac{M^3}{k}\left(1-e^{-2kb|\phi|}\right), \tag{2.53}$$

which depends weakly on b in the large kb limit. Since both M and k are of the Planck scale, so does M_{Pl}.

The physical masses on the visible brane can be determined by considering a Higgs field H in the Lagrangian in the visible sector. The action of a Higgs field can be written as

$$\begin{aligned}
S_{\rm vis} &\supset \int dx^4 \sqrt{g_{\rm vis}} \left\{ g_{\rm vis}^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda \left(|H|^2 - v_0^2 \right)^2 \right\} \\
&= \int dx^4 \sqrt{\bar{g}} e^{-4kb\pi} \left\{ \bar{g}^{\mu\nu} e^{2kb\pi} D_\mu H^\dagger D_\nu H - \lambda \left(|H|^2 - v_0^2 \right)^2 \right\} \\
&\to \int dx^4 \sqrt{\bar{g}} \left\{ \bar{g}^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda \left(|H|^2 - e^{-2kb\pi} v_0^2 \right)^2 \right\}, \qquad (2.54)
\end{aligned}$$

where λ is the coupling constant, and the last line is found by the wave function renormalization $H \to e^{kb\pi} H$. In this action, v_0 is a mass parameter. The last equation shows that the physical mass scale is set by the symmetry-breaking scale $v \equiv v_0 e^{-kb\pi}$. In other words, if $v_0 \sim 10^{19}$GeV is the Planck mass, then the corresponding physical mass is given by $v = v_0 e^{-kb\pi} \sim$TeV, provided that $kb \sim 10$. Thus, because of the exponential factor, a large suppression of the mass scales can be generated although the hierarchies among the parameters k, $1/b$, and M are not very large.

Kaluza-Klein modes

The Kaluza-Klein modes of a bulk field in the Randall-Sundrum scenario were found by [41]. We consider a free scalar field whose action is given by

$$S = \frac{1}{2} \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{G} \left(G^{AB} \partial_A \Phi \partial_B \Phi - m^2 \Phi^2 \right). \quad (A,B = 0,1,2,3,4) \qquad (2.55)$$

Integrating the action by parts and inserting the RS metric into the action, one gets

$$S = \frac{1}{2} \int d^4x \int_{-\pi}^{\pi} bd\phi \left\{ e^{-2kb|\phi|} \partial_A \Phi \partial^A \Phi + \frac{\Phi}{b^2} \partial_4 \left(e^{-4kb|\phi|} \partial_4 \Phi \right) - m^2 e^{-4kb|\phi|} \Phi \right\}. \qquad (2.56)$$

Since the RS metric is Poincaré invariant everywhere in the ordinary dimensions, one can expand the modes into the form:

$$\Phi(x^\mu, \phi) = \sum_n \frac{e^{2kb|\phi|}}{N_n} \psi_n(x^\mu) f_n \left(\frac{m_n}{k} e^{kb|\phi|} \right), \qquad (2.57)$$

where f_n and m_n are the eigenfunctions and the corresponding eigenvalues respectively of the following equation

$$z_n^2 \frac{\partial^2 f_n}{\partial z_n^2} + z_n \frac{\partial f_n}{\partial z_n} + \left[z_n^2 - \left(4 + \frac{m^2}{k^2} \right) \right] f_n = 0, \qquad (2.58)$$

where $z_n = m_n e^{kb|\phi|}/k$. N_n is the normalization constant for $f_n(z_n)$. The solution of $f_n(z_n)$ is given by the Bessel function of order $\alpha = \sqrt{4+m^2/k^2}$:

$$f_n(z_n) = J_\alpha(z_n) + b_{n\alpha} Y_\alpha(z_n). \tag{2.59}$$

Here both m_n and $b_{n\alpha}$ will be determined by boundary conditions, where the derivatives of $f_n(z_n)$ at the orbifold fix points are continuous, as required by the $\mathbb{Z}_2$ symmetry. In particular, when $e^{kb\pi} \gg 1$, m_n is determined by the equation

$$2J'_\alpha\left(\frac{m_n}{k}e^{kb\pi}\right) + \frac{m_n}{k}e^{kb\pi} J_\alpha\left(\frac{m_n}{k}e^{kb\pi}\right) = 0. \tag{2.60}$$

It follows that the lightest modes have $m_1 e^{kb\pi}/k$ of order unity, and that the masses of the Kaluza-Klein modes are suppressed exponentially with respect to m. If m_1 is of order of the Planck scale 10^{19} GeV and kb around 10, then $m_1 e^{kb\pi}/k \sim 1$ TeV, consistent with Randall and Sundrum's idea. This mass suppression is due to the warp factor $e^{2kb|\phi|}$ in the eigenexpansion (2.57), which has the greatest value at $\phi = \pi$, and thus the mass eigenmodes behave as if they are localized on the TeV-brane. Furthermore, if $m = 0$, there exists a massless mode where $m_{12} = 0$, which is usually identified as the massless graviton. Other non-zero solutions of m_{n2} can be treated as the spectrum of massive gravitons.

Radius Stabilization

In the discussion of graviton solutions in the metric (2.52), we have actually omitted a modulus field, whose vacuum expectation is b, the radius of the extra dimension. Since the solution of the Hierarchy Problem depends on b, the modulus field may have significant effects. The complete metric describing gravitons and modulus field in the Randall-Sundrum scenario is

$$d\hat{s}^2 = \overline{G}_{AB}dx^A dx^B = e^{-2kb(x^\gamma)|\phi|}\overline{g}(x^\gamma)dx^\mu dx^\nu - b^2(x^\gamma)d\phi^2. \tag{2.61}$$

Then the effective $4D$ action containing the curvature terms becomes [42]

$$S_{gravity} \supset 2M^3 \int d^4x \int_{-\pi}^{\pi} d\phi e^{-2kb|\phi|} \left[\sqrt{\bar{g}}\bar{R}6k|\phi|\partial_\mu b \partial^\mu b - 6k^2|\phi|^2 b\partial_\mu b\partial^\mu b + b\bar{R} \right]$$
$$= 2\frac{M^3}{k} \int d^4x \left[\left(1 - \frac{k\varphi}{24M^3}\right)\right] \sqrt{\bar{g}}\bar{R} + \frac{1}{2}\int d^4x \sqrt{\bar{g}}\partial_\mu\varphi\partial^\mu\varphi,$$

where $\varphi = e^{k\pi b}\sqrt{24M^3/k}$. Thus the original Randall-Sundrum scenario contains a massless scalar which couples to matter in TeV energy scales, contradicting to experimental observations. Therefore, it requires a modulus stabilization in order that the RS metric really solves the Hierarchy Problem.

Goldberger and Wise [42, 43] proposed a solution by employing a scalar field with the bulk action (2.55)

$$S_\Phi = \frac{1}{2}\int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{G}\left(\overline{G}^{AB}\partial_A\Phi\partial_B\Phi - m^2\Phi^2\right).$$

The scalar field is required to interact with the hidden and visible branes via

$$S_{\text{hid}} = -\int d^4x \sqrt{g_{\text{hid}}}\lambda_{\text{hid}}(\Phi^2 - v_{\text{hid}}^2)^2,$$
$$S_{\text{vis}} = -\int d^4x \sqrt{g_{\text{vis}}}\lambda_{\text{vis}}(\Phi^2 - v_{\text{vis}}^2)^2,$$

where $\lambda_{\text{vis, hid}}$ and $v_{\text{vis, hid}}$ are constants with mass dimensions $3/2$ and -2 respectively. These interaction terms lead to an expectation value $\Phi(\phi)$ of Φ along the extra dimension, and $\Phi(\phi)$ has boundary conditions $\Phi(0) = v_{\text{hid}}$ and $\Phi(\pm\pi) = v_{\text{vis}}$. The solution of $\Phi(\phi)$ is given by

$$\Phi(\phi) = e^{2kb|\phi|}(Ae^{\nu kb|\phi|} + Be^{-\nu kb|\phi|}),$$

where $\nu = \sqrt{4 + m^2/k^2}$, and, in the limit of large $\lambda_{\text{vis, hid}}$,

$$A = v_{\text{vis}}e^{-(2+\nu)kb\pi} - v_{\text{hid}}e^{-2\nu kb\pi}, \qquad B = v_{\text{hid}} - A. \tag{2.62}$$

Putting this solution into the action and taking the limit $\varepsilon \equiv m^2/4k^2 \ll 1$, one finds the effective potential for φ

$$V_\varphi(\varphi) = \frac{k^3}{144M^6}\varphi^4\left[v_{\text{vis}} - v_{\text{hid}}\left(\frac{\varphi}{\sqrt{24M^3/k}}\right)^\varepsilon\right], \tag{2.63}$$

which has a minimum at

$$k\langle b\rangle = \frac{1}{\varepsilon\pi}\ln\frac{v_{\rm hid}}{v_{\rm vis}} = \frac{4k^2}{\pi m^2}\ln\frac{v_{\rm hid}}{v_{\rm vis}}. \tag{2.64}$$

As the logarithmic term is of order unity, one needs only $m/k \sim 10^{-1}$ to get $k\langle b\rangle \sim 10$, which would again solve the Hierarchy Problem.

Chapter 3

Review of Extra Forces

The existence of extra forces can be understood as a by-product of the conservation of $5D$ total momentum whence the $4D$ total momentum may not necessarily be conserved. This has interesting implications since these extra forces appears non-gravitational and the resulting $4D$ trajectories are no longer geodesics. In order to study their effects, we first give a review of the definition of forces in $4D$ theory. Then we proceed to find the extra forces in the Space-Time-Matter (STM) theory by means of dimension reduction to project the $5D$ geodesics onto the ordinary $4D$ dimensions. From this, we will see how the $4D$ rest mass of a particle varies with the fifth coordinate in the course of motion. Lastly, the extra forces in a general brane model are also studied.

3.1 Definition of Force in $4D$

In a four-dimensional Minkowski space, the motion of a test particle at a position x^μ is described by the four-velocity

$$u^\mu = \frac{dx^\mu}{ds}, \tag{3.1}$$

where s is the proper time and u^μ satisfies $u^\mu u_\mu = c^2$ or 1 if the natural unit $c \equiv 1$ is used. The four-momentum is defined as $p^\mu = m_0 u^\mu$, where m_0 is the $4D$ rest mass.

The four-force is then given by

$$F^{\mu} = \frac{dp^{\mu}}{ds} = \frac{d}{ds}(m_0 u^{\mu}) = m_0 \frac{du^{\mu}}{ds} + \frac{dm_0}{ds} u^{\mu}. \tag{3.2}$$

This has an immediate consequence that

$$F^{\mu} u_{\mu} = F_{\mu} u^{\mu} = \frac{dm_0}{ds}. \tag{3.3}$$

In the usual $4D$ theory, m_0 is a fundamental constant. Thus, $F^{\mu} = m_0(du^{\mu}/ds)$ and $F^{\mu} u_{\mu} = F_{\mu} u^{\mu} = 0$.

In a four dimensional curvillinear space, the appropriate definition of the four-force is given by

$$F^{\mu} = \frac{dp^{\mu}}{ds} = \frac{d}{ds}(m_0 u^{\mu}) = m_0 \frac{D^{(4)} u^{\mu}}{ds} + \frac{dm_0}{ds} u^{\mu}, \tag{3.4}$$

where $D^{(4)}/ds$ is the $4D$ covariant derivative satisfying that

$$\frac{D^{(4)} g_{\mu\nu}}{ds} = 0, \tag{3.5}$$

and we get the same product of index contraction:

$$F^{\mu} = \frac{dp^{\mu}}{ds} = \frac{d}{ds}(m_0 u^{\mu}) = m_0 \frac{D^{(4)} u^{\mu}}{ds} + \frac{dm_0}{ds} u^{\mu}. \tag{3.6}$$

3.2 Extra Forces in STM Theory

The properties of extra forces for a $5D$ massive particle in the Space-Time-Matter theory were first examined by Mashhoon, Wesson and Liu [15]. In the following, we will work with the canonical metric (2.35):

$$d\hat{s}^2 = \frac{l^2}{L^2} g_{\alpha\beta}(x^{\gamma}, l) dx^{\alpha} dx^{\beta} - dl^2. \tag{2.35}$$

A Lagrangian can be defined from the metric (2.35) via

$$\mathscr{L} = \frac{d\hat{s}}{d\lambda} = \left[\frac{l^2}{L^2} g_{\alpha\beta}(x^{\gamma}, l) \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} - \left(\frac{dl}{d\lambda} \right)^2 \right]^{\frac{1}{2}}, \tag{3.7}$$

where λ is an arbitrary affine parameter.

The equation of motion can be obtained by varying the Lagrangian over the affine parameter λ. The 4D momenta in the ordinary dimensions and the respective equations of motion are given by

$$\begin{aligned} \hat{p}_\alpha &= \frac{\partial \mathscr{L}}{\partial u^\alpha} = \frac{l^2}{L^2}\frac{g_{\alpha\beta}(x^\gamma,l)u^\beta}{\mathscr{L}}, \\ \frac{d\hat{p}_\alpha}{d\lambda} &= \frac{\partial \mathscr{L}}{\partial x^\alpha} = \frac{1}{2\mathscr{L}}\left(\frac{l^2}{L^2}\right)\frac{\partial g_{\sigma\beta}}{\partial x^\alpha}u^\sigma u^\beta. \end{aligned} \tag{3.8}$$

Similarly, the extra momentum and the respective equation of motion are

$$\begin{aligned} \hat{p}_l &= -\frac{\partial \mathscr{L}}{\partial (dl/d\lambda)} = -\frac{1}{\mathscr{L}}\frac{dl}{d\lambda}, \\ \frac{d\hat{p}_l}{d\lambda} &= \frac{\partial \mathscr{L}}{\partial l} = \frac{1}{\mathscr{L}}\left(\frac{l}{L^2}\right)\left(g_{\sigma\beta}+\frac{l}{2}\frac{\partial g_{\sigma\beta}}{\partial l}\right)\frac{dx^\sigma}{d\lambda}\frac{dx^\beta}{d\lambda}. \end{aligned} \tag{3.9}$$

Eqs. (3.8) and (3.9) together give

$$\frac{du^\alpha}{d\lambda}+\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = -g^{\alpha\beta}\frac{\partial g_{\gamma\beta}}{dl}\frac{dl}{d\lambda}u^\gamma+\left[\frac{1}{\mathscr{L}}\frac{d\mathscr{L}}{d\lambda}-\frac{2}{l}\frac{dl}{d\lambda}\right]u^\alpha, \tag{3.10}$$

$$\frac{d^2l}{d\lambda^2}-\frac{1}{\mathscr{L}}\frac{d\mathscr{L}}{d\lambda}\frac{dl}{d\lambda} = -\frac{l}{L^2}\left(g_{\sigma\beta}+\frac{l}{2}\frac{\partial g_{\sigma\beta}}{\partial l}\right)\frac{dx^\sigma}{d\lambda}\frac{dx^\beta}{d\lambda}. \tag{3.11}$$

Here $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol of the second kind made out of $g_{\alpha\beta}(x^\gamma,l)$:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(\frac{\partial g_{\beta\sigma}}{\partial x^\alpha}+\frac{\partial g_{\gamma\sigma}}{\partial x^\beta}-\frac{\partial g_{\beta\gamma}}{\partial x^\sigma}\right).$$

If λ is chosen to be the 4D proper time s, so that $g_{\alpha\beta}u^\alpha u^\beta = 1$, then the set of equations of motion becomes

$$\frac{du^\alpha}{ds}+\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = -g^{\alpha\beta}\frac{\partial g_{\gamma\beta}}{dl}\frac{dl}{ds}u^\gamma+\left[\frac{1}{\mathscr{L}}\frac{d\mathscr{L}}{ds}-\frac{2}{l}\frac{dl}{ds}\right]u^\alpha, \tag{3.12}$$

$$\frac{d^2l}{ds^2}-\frac{1}{\mathscr{L}}\frac{d\mathscr{L}}{ds}\frac{dl}{ds} = -\frac{l}{L^2}-\frac{l^2}{2L^2}\frac{\partial g_{\sigma\beta}}{\partial l}\frac{dx^\sigma}{ds}\frac{dx^\beta}{ds}. \tag{3.13}$$

Under the parametrization of s,

$$\mathscr{L}^2 = \frac{l^2}{L^2}-\left(\frac{dl}{ds}\right)^2. \tag{3.14}$$

Then Eq. (3.13) can be rewritten as

$$\frac{d^2 l}{ds^2} - \frac{1}{l}\left(\frac{dl}{ds}\right)^2 = -\left[\frac{l^2}{L^2} - \left(\frac{dl}{ds}\right)^2\right]\left[\frac{1}{l} + \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial l} u^\beta u^\gamma\right]. \tag{3.15}$$

Using Eqs. (3.14) and (3.15), one finds from Eq. (3.12) that

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = \left(-g^{\alpha\sigma} + \frac{1}{2} u^\alpha u^\sigma\right) \frac{\partial g_{\sigma\beta}}{\partial l} \frac{dl}{ds} u^\beta. \tag{3.16}$$

Thus the extra force per unit mass acting on a 5D massive particle in the canonical metric can be defined as

$$f^\alpha = \left(-g^{\alpha\sigma} + \frac{1}{2} u^\alpha u^\sigma\right) \frac{\partial g_{\sigma\beta}}{\partial l} \frac{dl}{ds} u^\beta. \tag{3.17}$$

The extra force is non-zero provided that the 4D metric depends on the extra coordinate and there is motion in the extra dimension. The extra force defined in Eq. (3.17) is shown to be completely general, and it can also be derived from the perturbation analysis on the 4D metric in the extra coordinate l [15]. Ponce de Leon later gives a more concrete definition of an extra force and applies it to 5D massless particle [44].

Eq. (3.17) can be decomposed into two components: $f^\mu = f^\mu_\parallel + f^\mu_\perp$, where

$$f^\mu_\perp = (-g^{\alpha\sigma} + u^\mu u^\sigma) \frac{\partial g_{\sigma\beta}}{\partial l} \frac{dl}{ds} u^\beta, \tag{3.18}$$

$$f^\mu_\parallel = -\frac{1}{2} u^\mu u^\sigma \frac{\partial g_{\sigma\beta}}{\partial l} \frac{dl}{ds} u^\beta, \tag{3.19}$$

such that

$$f^\mu_\parallel u_\mu = -\frac{1}{2} u^\sigma \frac{\partial g_{\sigma\beta}}{\partial l} \frac{dl}{ds} u^\beta \neq 0 \tag{3.20}$$

and

$$f^\mu_\perp u_\mu = 0. \tag{3.21}$$

In other words, $f^\mu_\perp$ is the component normal to the 4-velocity u^μ, which could be due to ordinary four-dimensional forces; $f^\mu_\parallel$ is the component parallel to u^μ and there exists no 4D analogue. Thus $f^\mu_\parallel$ is the "anomalous" fifth force that would lead to anomalous effects in 4*D* observations. For instance, by comparing Eq. (3.20) to Eq. (3.6), an 4*D* observer would conclude that the 4*D* rest mass is varying with a rate given by

$$\frac{1}{m_0}\frac{dm_0}{ds} = f_{\parallel}^{\mu} u_\mu = -\frac{1}{2} u^\sigma \frac{\partial g_{\sigma\beta}}{\partial l}\frac{dl}{ds} u^\beta. \tag{3.22}$$

3.3 Variation of 4*D* Mass

As discussed, the existence of the extra forces might imply that the 4*D* rest mass, which should be a fundamental constant in 4*D* theories, should appear changing with the proper time.

To see the effect of the anomalous fifth force, we will assume that a bulk particle has a constant bulk mass $M_5 > 0$, and we define a five dimensional momentum as

$$P^A = M_5 U^A = M_5 \left(\frac{dx^\mu}{d\hat{s}}, \frac{dl}{d\hat{s}}\right), \qquad (A = 0,1,2,3,4) \tag{3.23}$$

where the five-velocity $U^A = (dx^\mu/d\hat{s}, dl/d\hat{s})$ is a direct extension of the four-velocity. By definition, $U^A U_A = c^2 \equiv 1$, and therefore

$$P^A P_A = M_5^2, \tag{3.24}$$

where raising and lowering indices in 5*D* are done by the 5*D* canonical metric $\hat{g}_{AB}$.

A 4*D* observe would have defined a 4*D* rest mass by the relation

$$p^\mu p_\mu = m_0^2 u^\mu u_\mu = m_0^2. \tag{3.25}$$

Here, raising and lowering indices are done by the 4*D* metric $g_{\mu\nu}$. Since the canonical metric does not have a cross term involving both the 4*D* and the extra coordinates, $p^\mu = P^\mu$ and $p_\mu = P_\mu$.

Eqs. (3.24) and (3.25) imply that

$$m_0^2 + \frac{l^2}{L^2} P^4 P_4 = \frac{l^2}{L^2} M_5^2. \tag{3.26}$$

Rewriting this equation in terms of the $4D$ line element ds, one gets the relation between the $4D$ and $5D$ masses:

$$m_0 = \frac{l}{L} M_5 \left[1 - \frac{L^2}{l^2} \left(\frac{dl}{ds} \right)^2 \right]^{-\frac{1}{2}}. \tag{3.27}$$

We see that the variation of the $4D$ rest mass is related to the motion in the extra dimension.

If $M_5 = 0$, then

$$d\hat{s}^2 = 0 \qquad \text{or} \qquad \frac{l^2}{L^2} ds^2 = dl^2. \tag{3.28}$$

Furthermore, we can define the $4D$ rest mass as

$$m_0 = \frac{l}{L} P_4 = \frac{l}{L} \frac{dl}{d\lambda}, \tag{3.29}$$

where λ is an affine parameter. Therefore, a $5D$ massless test particle may appear massive in four dimensions provided that it is non stationary in the extra dimension.

Eqs. (3.27) and (3.29) are the classical analogues of the Kaluza-Klein modes of extra dimensional quantum fields (e.g. Eqs. (2.44) and (2.60)). Whereas the Kaluza-Klein modes consist of discrete values of mass eigenvalues, the variations of $4D$ rest masses (as a result of the extra forces) are continuous. One can also find the Kaluza-Klein modes in STM theory, or, on the contrary, find the variations of $4D$ rest masses in ADD and RS-I models. Indeed, the latter will be applied in Chapter 7 to generate a mass hierarchy between particles with different trajectories in the same $5D$ world.

3.4 Position-Momentum Uncertainty Relation

Perhaps the existence of extra forces might lead to deviations of trajectories that cannot be explained by any existing $4D$ theories. These deviations could have been treated in the laboratory as experimental errors, and their significance could have been totally neglected. Of course, since the theory developed so far must be deterministic, we can differentiate the effects of extra forces from what is really experiment errors. But that is not enough! As scientists are always being crazy, we even want to ask: can the existence of the extra forces explain the Heisenberg uncertainty relations, at least in the coordinate representation?

Since the invention of the matrix mechanics by Heisenberg, the non-commutative mathematical structure has been the most important building block of quantum mechanics and field theory. The immediate consequence of non-commutative operations is that observables (measurable quantities represented by operators), say the coordinate x and the respective momentum p, no longer have definite values but must be statistical distributed over some mean values $\langle x\rangle$ and $\langle p\rangle$ with some standard deviations $\langle\Delta x\rangle$ and $\langle\Delta p\rangle$, respectively. The Heisenberg uncertainty relation says because x and p are non-commutative (complementary variables), the products of the standard deviations must be bounded below [45]:

$$\langle\Delta x\rangle\langle\Delta p\rangle \geq \frac{\hbar}{2}. \tag{3.30}$$

It seems that the Heisenberg uncertainty relation can only be understood in the quantum theory. The first attempt in understanding the quantum phenomenon of the electron with a $5D$ classical wave equation was made by Klein [2]. He showed that if the fifth dimension is periodic (not necessarily compactified), and if the second derivative with respect to the fifth coordinate is interpreted as the momentum of the electron, then he was able to obtain the Scrödinger equation for an electron moving in a $5D$ Kaluza-Klein world. The Planck constant is then related to the period of the extra dimension. Recently, Wesson [28] considers a $5D$ massless particle ($M_5 = 0$) under the following metric:

$$d\hat{s}^2 = \frac{L^2}{l^2} g_{\alpha\beta}(x^\gamma, l)dx^\alpha dx^\beta - \frac{L^4}{l^2}dl^2, \tag{3.31}$$

which can be obtained from the canonical metric by the change of variable $l \to L/l$. He considers the dot product $dp_\alpha dx^\alpha$ ($\alpha = 0,1,2,3$), where $p_\alpha = mu^\alpha = mdx^\alpha/ds$

with m being the $4D$ rest mass and s being the $4D$ proper time. Instead of using the definition (3.29), Wesson relates the $4D$ rest mass to the fifth coordinate through Compton's relation:

$$l = \frac{h}{m}, \tag{3.32}$$

where h is the Planck constant, and we have set $c \equiv 1$. Then the differential dp_α becomes

$$dp_\alpha = m du_\alpha - u_\alpha \frac{h}{l^2} dl, \tag{3.33}$$

If the $4D$ acceleration vanishes, then $du_\alpha dx^\alpha = 0$, and the absolute value of the dot product $dp_\alpha dx^\alpha$ becomes

$$|dp_\alpha dx^\alpha| = \frac{h}{l^2}|dl ds| = h\frac{dn_l^2}{n_l}, \tag{3.34}$$

where

$$n_l = \frac{L}{l}. \tag{3.35}$$

In the calculation, the identity $1 = u_\alpha u^\alpha = (L^2/l^2)u_l^2$ for $5D$ massless particle has been used. Thus, if $dn_l^2/n_l > 1/4\pi$, then we get the usual Heisenberg uncertainty relation. Wesson later [30] takes another route by transforming Milne's metric into a "metric wave", which is a metric whose coefficients in the $4D$ parts are complex waves, and he concludes that for a $5D$ massless particle, de Broglie's relations can be recovered if the fifth coordinate is identified as the induced $4D$ mass of the particle. Both Klein's and Wesson's work show that it may be possible to interpret quantum phenomena as some dynamical effects of higher dimensional motions.

3.5 Extra Forces in Brane Theories

The calculations of extra forces with the canonical metric in Section 3.2 can be easily extended to brane theories. The primitive form of a warped extra dimensional world can be described by the following metric [18, 23]:

$$d\hat{s}^2 = w(\phi) g_{\mu\nu}(x^\gamma) dx^\mu dx^\nu - d\phi^2, \tag{3.36}$$

where only the zero-th order of $g_{\mu\nu}$ has been considered; that is, we are working with foliated manifolds. Higher order terms of $g_{\mu\nu}$ may depend on ϕ. The equations of motion are

$$\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=-\frac{w'}{w}\frac{dx^\mu}{d\lambda}\frac{d\phi}{d\lambda}, \qquad (\mu=0,1,2,3) \tag{3.37}$$

$$\frac{d^2\phi}{d\lambda^2}=-\frac{w'}{2}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{3.38}$$

where $w'=dw/d\phi$, λ is an affine parameter and $\Gamma^\mu_{\beta\gamma}$ the Christoffel symbols of the second kind of $g_{\mu\nu}(x^\gamma)$. Since the ordinary dimensions are orthogonal to the fifth dimension, they together form a subspace and thus the $4D$ proper time $ds^2\equiv g_{\mu\nu}dx^\mu dx^\nu$ is an affine parameter. In other words, whether $\lambda=\hat{s}$ or s does not alter the equation of motion for x but it does for ϕ.

If $\lambda=\hat{s}$ and $g_{\mu\nu}=\eta_{\mu\nu}$ is the Minkowski metric, the equations of motion can be integrated once [18, 23]:

$$\frac{dx^\mu}{d\hat{s}}=\frac{u^\mu_{(5)}}{w(\phi)}, \tag{3.37'}$$

$$\frac{d\phi}{d\hat{s}}=\frac{\eta_{\mu\nu}u^\mu_{(5)}u^\nu_{(5)}}{w(\phi)}-1, \tag{3.38'}$$

where $u^\mu_{(5)}$ are integration constants.

If $\lambda=s$ and so $g_{\mu\nu}(dx^\mu/ds)(dx^\nu/ds)=1$ [15], then the Lagrangian is simply $\mathscr{L}^2=w-(d\phi/ds)^2$. The second equation of Eq.(3.38) becomes

$$\frac{d^2\phi}{ds^2}=-\frac{w'}{2}+\frac{w'}{w}\left(\frac{d\phi}{ds}\right)^2. \tag{3.38''}$$

Substituting Eq.(3.38”) into (3.37), one gets

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=0. \tag{3.37''}$$

Thus, there is no extra force under the parametrization s. This agrees with the result obtained when the canonical metric is employed if the $4D$ metric is independent of the extra coordinate [15, 12].

It is desirable to express the proper-time derivatives into the coordinate-time derivatives. Indeed, in the laboratory, all measurements are conducted in the coordinate-time. Any observable effect of the extra dimension would thus evolve in coordinate-time t instead of the proper-time s. For massive particles, $dt/ds > 0$ (for massive anti-particle, $dt/ds < 0$ [46]). In other words, t is monotonic increasing function of s and thus possesses an inverse function $s(t)$. Therefore, the dynamical variables x, y, z and ϕ can be rewritten as functions of t. Making use of the identities

$$\frac{d}{ds} = \frac{dt}{ds}\frac{d}{dt} \quad \text{and} \quad \frac{d^2}{ds^2} = \left(\frac{dt}{ds}\right)^2 \left(\frac{d}{dt}\right)^2 + \frac{d^2t}{ds^2}\frac{d}{dt},$$

one can transform Eq. (3.37”) into the form [33]

$$\ddot{x}^a + \Gamma^a_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma - \Gamma^t_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma \dot{x}^a = 0, \quad (a = 1, 2, 3) \tag{3.39}$$

where the dot denotes the coordinate time derivative, with $\dot{x}^t = 1$. This has the usual form of the equation of motion in the derivative of coordinate time, which means that no extra force could be detected when the dynamical variables are measured in coordinate time. Modifications of the metric (3.36) are therefore needed if we want to have extra forces measurable in the laboratory. There could be many ways of carrying out such modifications, which depend on how we construct the models.

Chapter 4

Extra Forces in the $5D$ Friedmann Universe

If extra dimensions really exist, then the position and the velocity, for example, of a particle that can be measured in an experiment are only the four dimensional projections of the corresponding quantities in the higher dimensional space. This opens up the idea that some kind of uncertainties in the projected 4*D* motion may exist due to the uncontrollable and unobservable motions in the extra dimensions. We have seen in Section 3.4 that there are indeed some attempts to explain quantum fluctuations in terms of 5*D* particle and wave dynamics. These theories are not complete but very instructive. It is one of the purposes of this thesis to see if fluctuations similar to those in the quantum theory (e.g. Eq. (3.30)) can be found in 5*D* particle trajectories.

Trajectories are determined by the metric of the underlying spacetime. Their effects on the projected motion in the ordinary space will be studied. As a start, a generalized verison of the ADD metric will be employed. The generalization, which is the Robertson-Walker metric in five-dimensional space-time, can be used to describe a higher-dimensional Friedmann universe. The Friedmann universe can be closed or open, and the scales of the spatial dimensions may depend on time. The deviations of the projected 4*D* trajectoies with respect to those in the 4*D* theory will be studied in this chapter, and the results will be compared to the classic results in the quantum theory.

As with the authors of previous literatures, we will assume that the $5D$ particle can be detected by an $4D$ observer, regardless of its position in the extra dimension [15, 12, 31, 32, 21, 44]. This is very different from the usual treatment in the brane theory, where the observer can only detect matters on the visible brane. However, it is this assumption that leads to interesting pheonomenology.

4.1 Generalized Robertson-Walker Metric

The Robertson-Walker metric generalized to a higher dimensional spacetime is given by:

$$d\hat{S}^2 = dt^2 - a^2(t)\left(\frac{dr_a^2}{1-k_a r_a^2} + r_a^2 d\Omega_a^2\right) - b^2(t)\left(\frac{dr_b^2}{1-k_b r_b^2} + r_b^2 d\Omega_b^2\right), \tag{4.1}$$

where k_a and k_b are curvature parameters and can only take the values -1, 0 and 1. The variables r's and Ω's are dimensionless and the scale factors $a(t)$ and $b(t)$ carry the sizes of the ordinary and the extra dimension respectively. This is a cosmological viable metric. It leads to a set of generalized Friedmann equations which describe the evolutions of the scale factors $a(t)$ and $b(t)$. When one tries to write this generalized set of equations into a form which resembles the Friedmann equations in vacuum 3+1 dimensions, the terms containing k_b, $b(t)$ and $\dot{b}(t)$ together appear as the cosmological constant. In this way, rather than putting in it by hand as Einstein did, the cosmological constant automatically emerges and hence governs the behavior of $a(t)$. On the other hand, the observed current value of the cosmological constant becomes a direct probe of the extra dimension. The study of the effects of the metric (4.1) on the evolution of $a(t)$ in vacuum and its signature on the cosmological constant has been reviewed by several authors [47, 48, 49]. There have been some interesting results concerning the evolution of a flat $4D$ universe due to curved spatial extra dimenions.

When k_a and k_b are set to zero, and $a(t)$ and $b(t)$ are constant in time, the metric

(4.1) reduces to a 5D ADD metric.

Equations of Motion with One Extra Dimension

The equations of motion will be derived from Eq. (4.1) in this section. For simplicity, one ordinary and one extra dimension will be assumed in this section. Although, in general, the scale factors are governed by the (generalized) Friedmann equations, they will be assumed to take several forms that are appropriate for the present analysis. In this way, the scale factors could be canonically chosen to illustrate the general idea.

The equations of motion are given by

$$\begin{aligned} \frac{d^2t}{d\hat{S}^2} + \frac{a(t)\dot{a}(t)}{1-k_a r_a^2}\left(\frac{dr_a}{d\hat{S}}\right)^2 + \frac{b(t)\dot{b}(t)}{1-k_b r_b^2}\left(\frac{dr_b}{d\hat{S}}\right)^2 &= 0, \\ \frac{d^2r_a}{d\hat{S}^2} + 2\frac{\dot{a}(t)}{a(t)}\frac{dr_a}{d\hat{S}}\frac{dt}{d\hat{S}} + \frac{k_a r_a}{1-k_a r_a^2}\left(\frac{dr_a}{d\hat{S}}\right)^2 &= 0, \\ \frac{d^2r_b}{d\hat{S}^2} + 2\frac{\dot{b}(t)}{b(t)}\frac{dr_b}{d\hat{S}}\frac{dt}{d\hat{S}} + \frac{k_b r_b}{1-k_b r_b^2}\left(\frac{dr_b}{d\hat{S}}\right)^2 &= 0, \end{aligned} \tag{4.2}$$

where the dot means a time derivative: $\dot{f} = df/dt$. It is desirable to express the above set of equations as functions of t, which is a measurable quantity in the lab frame. Making use of the identities

$$\frac{d}{d\hat{S}} = \frac{dt}{d\hat{S}}\frac{d}{dt}$$

and

$$\frac{d^2}{d\hat{S}^2} = \left(\frac{dt}{d\hat{S}}\right)^2\frac{d^2}{dt^2} + \frac{d^2t}{d\hat{S}^2}\frac{d}{dt},$$

and dividing the resulting set of equations by $(dt/d\hat{S})^2$ on both sides, the equations of motion become

$$\ddot{r}_a + 2\frac{\dot{a}}{a}\dot{r}_a + \frac{k_a r_a \dot{r}_a^2}{1-k_a r_a^2} - \frac{a\dot{a}\dot{r}_a^3}{1-k_a r_a^2} = \frac{b\dot{b}\dot{r}_a\dot{r}_b^2}{1-k_b r_b^2}, \tag{4.3}$$

$$\ddot{r}_b + 2\frac{\dot{b}}{b}\dot{r}_b + \frac{k_b r_b \dot{r}_b^2}{1-k_b r_b^2} - \frac{b\dot{b}\dot{r}_b^3}{1-k_b r_b^2} = \frac{a\dot{a}\dot{r}_a^2\dot{r}_b}{1-k_a r_a^2}. \tag{4.4}$$

The fact that these two equations are completely symmetric by the transformations $r_a \leftrightarrow r_b$, $a(t) \leftrightarrow b(t)$ and $k_a \leftrightarrow k_b$ should be obvious. The term on the right hand side of Eq. (4.3) is the relativistic corrections of r_a due to the motions in the extra dimension. If extra dimensions do not exist, then this term should be set to zero, and the ordinary Friedmann-Robertson-Walker universe is recovered. Therefore, this term is an “extra force”, which arises when the motion in the higher dimensional space-time is projected on the ordinary space-time. This illustrates the possibility of deviations from a trajectory in ordinary mechanics predicted from the classical theory without extra dimensions.

The extra force depends on several factors. If the extra dimension is not evolving, *i.e.* $\dot{b} = 0$, then the $4D$ Poincaré invariance is preserved in the generalized Robertson-Walker metric, and so no extra force can be found. Thus the evolution of the extra dimension serves as the coupling mechanism between the extra and ordinary dimensions, such that the momentum exchange between these dimensions becomes possible. Also, the particle must be non-stationary for a non-zero extra force. This is a frame-dependent extra force but poses no contradiction to the theory of relativity since the Poincaré invariance has been broken by the evolution of the extra dimension.

There are two major differences between the definitions of the extra force in the Space-Time-Matter (STM) theory and the present work. (See Chapter 3.) Firstly, two different projection spaces have been used: STM theory describes the trajectories as functions of the $4D$ line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (where $\mu, \nu = 0, 1, 2, 3$ and $g_{\mu\nu}$ satisfies the $4D$ Einstein equation) while the present work uses the coordinate time t. This fact leads to the second difference: non-vanishing extra force exists in STM theory only if $\partial g_{\mu\nu}/\partial x^4 \neq 0$, that is, physics in our $4D$ space-time does depend on the extra coordinate. In order to explain this strange behavior, STM theorists have to invoke Mach's Principle, which asserts that a local particle's motions must be influenced by the whole universe. This is not the case in the present work.

With the metric (4.1), for example, one of the requirements for a non-vanishing extra force is that the scale factor $b(t)$ must be changing with time. However, both theories require non-zero velocities in the extra dimension (that is, $\dot{r}_b \neq 0$ in the present work) in order to get a non-zero extra force.

Eq. (4.3) can be solved using a standard transformation:

$$r_{a,b} = \begin{cases} \sin\psi_{a,b}\,, \\ \psi_{a,b}\,, \\ \sinh\psi_{a,b} \end{cases} \quad \text{for} \quad k_{a,b} = \begin{cases} 1 \\ 0 \\ -1 \end{cases} . \tag{4.5}$$

Then Eq. (4.3) is transformed into

$$\ddot{\psi}_a + 2\frac{\dot{a}}{a}\dot{\psi}_a - a\dot{a}\dot{\psi}_a^3 = b\dot{b}\dot{\psi}_a\dot{\psi}_b^2 \quad \text{and} \quad \ddot{\psi}_b + 2\frac{\dot{b}}{b}\dot{\psi}_b - b\dot{b}\dot{\psi}_b^3 = a\dot{a}\dot{\psi}_a^2\dot{\psi}_b. \tag{4.6}$$

These equations have exactly the same form for $k_a = k_b = 0$. Therefore the space is conformally flat for all cases of k_a and k_b. Manipulation of these two equations implies $b^2\psi_b \propto a^2\psi_a$, which further gives the first order solutions of the equations:

$$\frac{1}{a^4\dot{\psi}_a^2} = \frac{1}{a^2} + \frac{c_1}{b^2} + c_2 \quad \text{and} \quad \frac{1}{b^4\dot{\psi}_b^2} = \frac{c_3}{a^2} + \frac{1}{b^2} + c_4. \tag{4.7}$$

The equation of motion for ψ_a is therefore

$$\ddot{\psi}_a + 2\frac{\dot{a}}{a}\dot{\psi}_a - a\dot{a}\dot{\psi}_a^3 = \frac{\dot{b}\dot{\psi}_a}{b + (c_3/a^2 + c_4)\,b^3}\,. \tag{4.8}$$

If $b(t)$ is decreasing with time, then the right hand side serves as a damping term which is increasing with time. As a result the particle decelerates to rest as time goes on. The reverse is true for $b(t)$ increasing with time.

The trajectory can be examined by obtaining a perturbation solution to (4.8). As far as the effect of the extra dimension is concerned, $\dot{a}$ can be taken as zero for simplicity, and we set $a(t) \equiv 1$. Then $\psi_a = r_a$. The physical velocity of ψ_b is $v_b = b\dot{\psi}_b$. In the short time limit $t^{-1} \gg H_b v_b^2$ where $H_b \equiv \dot{b}/b$, the trajectory in the ordinary dimension can be written as

$$r_a(t) \doteq r_{a0} + \dot{r}_{a0}t + \frac{1}{2}H_{b0}v_{b0}^2\dot{r}_{a0}t^2, \tag{4.9}$$

where the subscript zero denotes values at some reference time $t_0 \equiv 0$.

Discussion

The effect of the extra force on $\dot{r}_a(t)$ can be understood as follows. From Eq. 4.3, the extra force is given by

$$F_{\text{ex}} = b\dot{b}\dot{r}_a \frac{\dot{r}_b^2}{1-k_b r_b^2}. \tag{4.10}$$

This extra force is parallel to its ordinary velocity $\dot{r}_a(t)$. The extra force seems to be dependent on the choice of reference frames (where $\dot{r}_a(t)$ may be different) [31]. In the approximation up to first order of $\dot{b}$, F_{ex} can be evaluated by using Eq. (4.6) and by taking $\dot{r}_a$ as a constant since the whole space is conformally flat. The fraction is always non-negative. That means, whether the extra force causes acceleration or deceleration on the particle depends only on the sign of $\dot{b}$. Suppose $\dot{b} > 0$. Then, once the particle has a velocity $\dot{r}_a \neq 0$, it accelerates and moves away from the point $\dot{r}_a = 0$ for all r_a in the phase plane $(r_a, \dot{r}_a)$. Thus, the phase line $\dot{r}_a = 0$ is unstable. If $\dot{b} < 0$, the extra force has a negative sign, and a non-stationary particle (whose $\dot{r}_a \neq 0$) will decelerate and move towards the phase line $\dot{r}_a = 0$. Thus $\dot{r}_a = 0$ is a stable phase line. In any case, a linear trend (either acceleration or deceleration) will be seen in $\dot{r}_a(t)$, consistent with the above result.

As in cosmology, the ratio $\dot{b}/b$ will be assumed to be slowly varying and can be taken as a constant value H_b. Then Eq. (4.9) implies that a particle in a 5D spacetime must have travelled farther than a particle in a 4D space-time if $H_b > 0$, and vice versa. This conclusion solely depends on the sign of H_b and not other factors.

The most interesting case occurs when $H_b > 0$. In this case, the phase line $\dot{r}_a = 0$ is unstable, and we have to consider the evolution of the density functions in the phase space [50]. In particular, in the following section, we will assume that all initial conditions have distributions around some mean values. The standard deviations or uncertainties of these initial conditions form phase space volumes, and the evolution of these volumes will resemble the evolution of the wave packets in quantum mechanics.

4.2 Unobserved Motions in Extra Dimensions

In the laboratory, the motions in the extra dimension cannot be detected experimentally. Thus, the particle trajectories must start with some unknown initial conditions in the extra coordinates. In an experiment, every time when a particle is let go, it starts with some measurable inital values of r_{a0} and $\dot{r}_{a0}$ but with unknown ψ_{b0} and v_{b0}. Therefore, ψ_{b0} and v_{b0} are random variables. As we have seen from Eq. (4.9), different values of v_{b0} result in different trajectories $r_a(t)$. After the repetition of a number of experiments with identical initial settings of r_{a0} and $\dot{r}_{a0}$ being run for a fixed period of time, the experimenter acquires a probability distribution of r_a at the end of each experiment trial. Thus, in this extra dimensional scenario, it seems possible that uncertainties of $4D$ position-momentum measurements can be naturally generated due to unobserved extra dimensional motions.

In this section, we want to calculate the uncertainties of $4D$ position-momentum measurements in the $5D$ Friedmann universe. We first follow the line of thought in the previous paragraph to start with the cases where the initial values of r_{a0} and $\dot{r}_{a0}$ are fixed while v_{b0} is randomized. Since the extra dimension is conformally flat, the choice of ψ_{b0} does not affect the observed trajectories in the ordinary dimension, as suggested by Eq. (4.9). Therefore, the randomization of ψ_{b0} can be ignored. Then we also consider cases where the initial r_{a0} and $\dot{r}_{a0}$ are gaussianly distributed around some mean values. These correspond to the situation in the real world because the distributions of r_{a0} and $\dot{r}_{a0}$ must satisfy the Heisenberg uncertainty relation. In the next section, we will compare our results with the quantum mechanical results.

Case 1: $\Delta\dot{r}_a = 0$**,** $\Delta r_a = 0$ **and** $\Delta v_b \neq 0$

The assertion that the extra dimensional motion cannot be monitored implies that the quantity v_{b0} should be randomized. The randomization of ψ_{b0} is ignored since the extra dimension is conformally flat. We now propose a probability distribution of v_{b0}. To get this, we notice that the $4D$ rest mass m_0 is related to the $5D$ rest mass

M_5 by

$$m_0 = \frac{M_5}{\sqrt{1 - v_b^2}}, \tag{4.11}$$

where $v_b = b\dot{\psi}_b$ is the velocity along the extra dimension. Thus m_0 is a variable of v_b. In other words, the observed rest mass of the particle could vary from M_5 to infinity if v_b is allowed to take any values in the range $[-1, 1]$. This contradicts with our observations. Clearly, we need m_0 to be confined within some constant values. This amounts to confine v_b around some mean values. Since the extra dimension has $\mathbb{Z}_2$ symmetry, it is natural to assume that v_b is symmetrically distributed around zero, and the width of the distribution should be much lower than the speed of light. For simplicity, we assume that v_b is gaussianly distributed around zero with a width $\Delta v_{b0} \ll 1$:

$$P_{v_{b0}}(v_{b0}) = \frac{1}{\sqrt{2\pi\Delta v_{b0}^2}} \exp\left\{-\frac{v_{b0}^2}{2\Delta v_{b0}^2}\right\}. \tag{4.12}$$

Thus, we are only interested in non-relativistic motion in the extra dimension. Then the mean value of the $4D$ rest mass is

$$\begin{aligned}
\overline{m}_0 &= M_5 \int \frac{1}{\sqrt{1 - v_{b0}^2}} P_{v_{b0}} dv_{b0} \\
&\approx M_5 \int \left(1 + \frac{1}{2} v_{b0}^2\right) P_{v_{b0}} dv_{b0} \\
&\approx M_5 (1 + \frac{1}{2}\Delta v_{b0}^2),
\end{aligned} \tag{4.13}$$

and the uncertainty is

$$\begin{aligned}
\int \left(\frac{M_5}{1 - v_{b0}^2} - M_5\right)^2 P_{v_{b0}} dv_{b0} &\approx M_5 \int \left(\frac{v_{b0}^2}{2} + \frac{3v_{b0}^4}{8} + \cdots\right)^2 P_{v_{b0}} dv_{b0} \\
&\approx \sqrt{\frac{3}{4}} M_5 \Delta v_{b0}^2.
\end{aligned} \tag{4.14}$$

Figure 4.1: The distribution of displacement $f(r_a)$ due to the motion in the extra dimension in a spacetime with metric (4.1). The initial conditions are arbitrarily chosen: $r_{a0} = 0$, $\dot{r}_{a0} = 1$, $b_0 = -\dot{b}_0 = 1$, and $t = 0.1$, $\Delta v_{b0} = 0.1$.

The probability distribution function for r_a is given by

$$f(r_a) = \int P_{v_{b0}} dv_{b0} \delta(r_a - r_a(t)) \\ = \begin{cases} \frac{1}{\sqrt{\pi}\Delta v_{b0} t} \frac{\exp\{-\frac{r_a - r_{a0} - \dot{r}_{a0} t}{H_{b0}\dot{r}_{a0}\Delta v_{b0}^2 t^2}\}}{\sqrt{|H_{b0}\dot{r}_{a0}(r_a - r_{a0} - \dot{r}_{a0} t)|}}, & \text{if } |r_a| > |r_{a0} + \dot{r}_{a0} t|; \\ 0, & \text{otherwise.} \end{cases} \tag{4.15}$$

An example of $f(r_a)$ is shown in Fig. 4.1.

We can calculate the mean value and the uncertainty (standard deviation) of

$r_a(t)$, which are respectively

$$\begin{aligned}\overline{r}_a(t) &\doteq r_{a0} + \dot{r}_{a0}t + \frac{1}{2}\dot{r}_{a0}H_{b0}t^2 \int v_{b0}^2 P_{v_{b0}} dv_{b0} \\ &\doteq r_{a0} + \dot{r}_{a0}t\left(1 + \frac{\Delta v_{b0}^2}{2}H_{b0}t\right),\end{aligned} \tag{4.16}$$

$$\begin{aligned}\Delta r_a(t) &\doteq \left\{\int \Big(r_a(t) - \overline{r}_a(t)\Big)^2 P_{v_{b0}} dv_{b0}\right\}^{\frac{1}{2}} \\ &= \frac{\Delta v_{b0}^2}{\sqrt{2}} H_{b0}\dot{r}_{a0}t^2,\end{aligned} \tag{4.17}$$

where $t^{-1} \gg H_{b0}$ in the short time limit. Thus the mean position of the ensemble evolves quadratically in time, showing the influence of the extra force. The uncertainty of the position also grows quadratically with time, meaning that the ensemble expands over the 4D space-time. At $t = 0$, $\Delta r_a(t) = 0$ because the ensemble is assumed to start with identical r_{a0}.

Define the ordinary momentum as $p_a = m_0\dot{r}_a$. Then m_0 also contributes to the distribution of p_a. The average value and the uncertainty of the momentum are respectively

$$\begin{aligned}\overline{p}_a &= \int (m_0\dot{r}_a) P_{v_{b0}} dv_{b0} \doteq M_5\dot{r}_{a0}\left(1 + \Delta v_{b0}^2 H_{b0}t\right), \\ \Delta p_a^2 &= \int (p_a - \overline{p}_a)^2 P_{v_{b0}} dv_{b0} \doteq M_5^2 \Delta v_{b0}^4 \dot{r}_{a0}^2 \left(1 + 2H_{b0}^2 t^2\right),\end{aligned} \tag{4.18}$$

where $p_{a0} = M_5\dot{r}_{a0}$ and $\Delta p_{a0}^2 = M_5\Delta v_{b0}^2\dot{r}_{a0}$. These equalities are evaluated in the short time limit $\Delta v_{b0}^2 H_{b0}t \ll 1$. Unlike $\Delta\dot{r}_a$, at $t = 0$, Δp_a is non-zero. This is because the 4D mass itself has a distribution when v_{b0} is randomized. We can now calculate the product of uncertainties:

$$\Delta r_a^2(t)\Delta p_a^2(t) \doteq \frac{1}{2}(M_5 H_{b0}\Delta v_{b0}^2 \dot{r}_{a0}^2 t^2)^2, \tag{4.19}$$

where higher order terms in H_{b0} have been neglected. Again, at $t = 0$, this product vanishes because we assume that the particle starts at definite values of r_{a0} and $\dot{r}_{a0}$. But in the laboratory, the distributions of r_{a0} and $\dot{r}_{a0}$ must satisfy the Heisenberg uncertainty relation. In the following cases, we will realize this assumption step by step.

Case 2: $\Delta\dot{r}_a \neq 0$**,** $\Delta r_a = 0$ **and** $\Delta v_b \neq 0$

In the above discussion, the velocity in the $4D$ spacetime has been kept constant while the velocity in the extra dimension has been randomized so that the total velocity does not exceed the limiting speed. As we have discussed, the phase line $\dot{r}_a = 0$ is unstable, and we have to consider the evolution of some phase space volumes around this line. Thus, in the following, we will first consider the initial position r_{a0} being also randomized. Since we should have at least a bit of control on the experimental setup, it would be natural to assume that r_{a0} is Gaussianly distributed, by virtue of the central limit theorem.

$$P_{r_{a0}}(r_{a0}) = \frac{1}{\sqrt{2\pi\Delta r_a^2}} \exp\left\{ -\frac{(r_{a0} - \overline{r}_{a0})^2}{2\Delta r_a^2} \right\}. \tag{4.20}$$

Thus Eq. (4.9) gives the following dispersion relations:

$$\begin{aligned} \overline{r}_a(t) &\doteq \overline{r}_{a0} + \dot{r}_{a0} t \left(1 + \frac{\Delta v_{b0}^2}{2} H_{b0} t \right), \\ \Delta r_a^2(t) &\doteq \Delta \dot{r}_{a0}^2 t^2 \left(1 + \Delta v_{b0}^2 H_{b0} t \right) \qquad (H_b v_{b0}^2\, t \ll 1), \end{aligned} \tag{4.21}$$

and $\overline{p}_a$ and Δp_a are still given by Eq. (4.18).

Case 3: $\Delta\dot{r}_a \neq 0$**,** $\Delta r_a \neq 0$ **and** $\Delta v_b \neq 0$

Finally, we also assume that $\dot{r}_a$ is gaussianly distributed with some mean and standard deviation. The probability distributions of the three dynamical variables are written as

$$P_{r_{a0}}(r_{a0}) = \frac{1}{\sqrt{2\pi\Delta r_a^2}} \exp\left\{ -\frac{(r_{a0} - \overline{r}_{a0})^2}{2\Delta r_a^2} \right\}, \tag{4.20}$$

$$P_{\dot{r}_{a0}}(\dot{r}_{a0}) = \frac{1}{\sqrt{2\pi\Delta \dot{r}_a^2}} \exp\left\{ -\frac{(\dot{r}_{a0} - \overline{\dot{r}}_{a0})^2}{2\Delta \dot{r}_a^2} \right\}, \tag{4.22}$$

$$P_{v_{b0}}(v_{b0}) = \frac{1}{\sqrt{2\pi\Delta v_{b0}^2}} \exp\left\{ -\frac{v_{b0}^2}{2\Delta v_{b0}^2} \right\}. \qquad (\Delta v_{b0} \ll \pi) \tag{4.12}$$

Figure 4.2: Spreads of particle motions described by Eq. (4.21) with three different values of $\dot{b}_0$: $\dot{b}_0 = 0$ (dotted), $\dot{b}_0 = -1 < 0$ (narrow) and $\dot{b}_0 = 1 > 0$ (thick). In all cases, $\bar{\dot{r}}_{a0} = 0$, $\Delta\dot{r}_{a0} = 0.2$, $t = 0.1$ and $b_0 = 1$. The sample size is 2000.

Again, ψ_b needs not to be considered since the extra dimension is homogeneous. The uncertainties of the distributions of r_a and $\dot{r}_a$ respectively can be easily calculated from these probability distributions:

$$\begin{aligned} \overline{r}_a &\doteq \overline{r}_{a0} + \bar{\dot{r}}_{a0} t + \frac{1}{2} H_{b0} \Delta v_{b0}^2 t^2 \\ \Delta {r_a}^2 &\doteq \Delta r_{a0}^2 + \Delta \dot{r}_{a0}^2 t^2 \left(1 + \Delta v_{b0}^2 H_{b0} t\right); \\ \overline{p}_a &\doteq M_5 \bar{\dot{r}}_a \left(1 + \frac{\Delta v_{b0}^2}{2} + \Delta v_{b0}^2 H_{b0} t\right) \doteq \overline{m}_0 \bar{\dot{r}}_a \left(1 + \Delta v_{b0}^2 H_{b0} t\right) \\ \Delta {p_a}^2 &\doteq \Delta p_{a0}^2 \left(1 + 2 H_{b0} \Delta v_{b0}^2 t\right), \end{aligned} \tag{4.23}$$

where $\Delta p_{a0}^2 \equiv \overline{m}_0^2 \Delta \dot{r}_{a0}^2 = M_5^2 (1 + \Delta v_{b0}^2) \Delta \dot{r}_{a0}^2$ and $H_b \Delta v_{b0}^2 t \ll 1$. In the expression for $\Delta {r_a}^2$, the last term in the bracket is due to the randomization of v_{b0}, while the other two terms come from the randomizations of r_{a0} and $\dot{r}_{a0}$. Similarly, in the bracket of the expression for $\Delta {p_a}^2$, the first term is due to the gaussian distribution of $\dot{r}_{a0}$, the second term comes from the randomization of the extra term in $r_a(t)$ (Eq. (4.9)). The factor $(1 + \Delta v_{b0}^4)$ in Δp_{a0}^2 comes from the randomization of the $4D$ rest mass m_0.

The product $\Delta r_a^2 \Delta p_a^2$ is thus given by

$$\Delta r_a^2 \Delta p_a^2 \doteq h_0^2 + \frac{\Delta p_{a0}^4}{\overline{m}_0^2} t^2 + \Delta v_{b0}^2 H_{b0} t \left(2 h_0^2 + \frac{3 \Delta p_{a0}^4}{\overline{m}_0^2} t^2\right), \tag{4.24}$$

where $h_0 \equiv \Delta r_{a0} \Delta p_{a0}$, and we have replaced M_5 by $\overline{m}_0 = M_5 (1 + \Delta v_{b0}^2 / 2)$ and retain terms of order up to Δv_{b0}^4.

We have assumed that v_b is peaked at zero. If v_b is peaked at other non-zero values, then the averaged $4D$ rest mass is modified as

$$\overline{m}_0 = \frac{M_5}{\sqrt{1 - \bar{v}_{b0^2}}} (1 + \frac{1}{2} \Delta v_{b0}^2). \tag{4.25}$$

Thus Eq. (4.24) still holds with $\overline{m}_0$ replaced by this modified expression.

Comparison with Quantum Mechanical Results

Eq. (4.24) resembles the Heisenberg uncertainty relation in quantum mechanics. Thus the probability distributions Eq. (4.20) and Eq. (4.22) can be interpreted as the consequence of quantum fluctuations. Furthermore, the product $h_0 \equiv \Delta r_{a0} \Delta p_{a0}$ of the distribution of the initial conditions must not vanish.

We want to compare Eq. (4.24) with that in quantum theory. Suppose that at time $t = 0$ a quantum particle is described by a one dimensional Gaussian wave packet $\phi(p_x)$ in the momentum space,

$$\phi(p_x) = \frac{1}{\pi^{1/4}\sqrt{\Delta p_{x0}}} \exp\left[-\frac{(p_x - p_0)^2}{2(\Delta p_{x0})^2}\right]. \tag{4.26}$$

Then the wave function in the configuration space can be derived from the Fourier transform of $\phi(p_x)$, which is also a Gaussian wave packet with $\Delta x_0 = \hbar/\Delta p_{x0}$. According to Schrödinger's equation, the uncertainty relation evolves as

$$\Delta x^2 \Delta p_x^2 = \hbar^2 + \frac{\Delta p_{x0}^4}{m^2} t^2. \tag{4.27}$$

It is a standard exercise to derive Eq. (4.27) in many elementary textbooks of quantum mechanics ([45], for example).

We can also obtain Eq. (4.27) from classical mechanics. From Newtonain mechanics, the trajectory of a free particle is described by

$$x(t) = x_0 + \dot{x}_0 t = x_0 + \frac{p_{x0}}{m} t, \tag{4.28}$$

where x_0 and $\dot{x}_0$ are the initial position and velocity respectively, and $p_{x0} = m\dot{x}_0$ is the initial momentum. Suppose that x_0 and $\dot{x}_0$ are normally distributed with variances Δx_0^2 and $\Delta \dot{x}_0^2$ respectively, then they simply add to give the variance of $x(t)$

$$\Delta x(t)^2 = \Delta x_0^2 + \Delta \dot{x}_0^2 t^2 = \Delta x_0^2 + \frac{\Delta p_{x0}^2}{m^2} t^2. \tag{4.29}$$

Since $\Delta p_x(t)^2 = \Delta p_{x0}^2$, its product with $\Delta x(t)^2$ reproduces Eq. (4.27) if $\Delta \dot{x}_0^2 \Delta p_{x0}^2 = \hbar^2$. Therefore, a wave packet from the classical trajectories of a free particle can also be produced if the initial conditions are randomized. The constant $\hbar$ is required to be non-zero by the virtue of Heisenberg uncertainty principle. This equation serves as a classical test of the current theory if we compare evolution of a wave packet in 5D with Eq. (4.27).

Eq. (4.24) is an example of such a wave packet in a $5D$ space-time. It is analogous to Eq. (4.27). The constant h_0 is required to be non-zero since we are considering the phase space volumes around the unstable phase line $\dot{r}_a$. Thus, from Eq. (4.24), we get two correction terms: one grows linearly with time, and the other grows cubically. These corrections terms are due to the evolution of the extra dimension and the random motions in it. For both terms, the coefficients are proportional to H_b, which is of cosmological time scale. Thus, Eq. (4.24) describes the same evolution of the width of a wave packet as that in the ordinary $4D$ space-time in the short time limit.

The parameters Δv_{b0} and H_{b0} can be measured as follows. We measure the spread of the wave packet of a free particle. From the time evolution of the wave packet, we get the coefficients of t- and t^3-dependent terms in the width of the wave packet, both of which are absent in quantum mechanics. Denote them as C_1 and C_3 respectively. Then Δv_{b0} and H_{b0} can be solved from the coupled set of equations:

$$C_1 = 2h_0^2 \Delta v_{b0}^2 H_{b0}, \quad C_3 = \Delta v_{b0}^2 H_{b0} \frac{3\Delta p_{a0}^4}{\overline{m}_0^2},$$

provided the initial values Δp_{a0} are obtained at the beginning of the experiment. Obviously, we identify $\overline{m}_0$ as the $4D$ rest mass (m in Eq. (4.27)) of the particle.

4.3 Equations of Motion with Three Extra Spatial Dimensions

The above model is relatively simple because only one extra dimension is assumed, which results in a conformally flat space. However, it has been shown that the evolution of $a(t)$ is coupled to that of $b(t)$ only if the number of extra dimensions is not less than three [47, 48, 49]. The resultant equations of motion would be considerably more complicated when more extra dimensions are employed. Some simplifications will be made.

It has been concluded that cosmological observations can be modelled with $k_a = 0$ and $k_b = 1$ [49]. This has a profound implication that the universe should appear flat while the current state of expansion is due to the curvature of the extra dimensions. In what follows, $k_a = 0$ will also be assumed.

The FRW metric is given by

$$d\hat{S}^2 = dt^2 - a^2(t)dr_a^2 - b^2(t)\Big(\frac{dr_b^2}{1-k_b r_b^2} + r_b^2 d\theta_b^2 + r_b^2 \sin^2\theta_b d\phi_b^2\Big). \tag{4.1'}$$

The corresponding equations of motion are

$$\begin{aligned} \ddot{r}_a + 2\frac{\dot{a}}{a}\dot{r}_a &= F_{ex}(t)\dot{r}_a\,, \\ \ddot{r}_b + 2\frac{\dot{b}}{b}\dot{r}_b + \frac{k_b r_b \dot{r}_b^2}{1-k_b r_b^2} - r_b\left(1-k_b r_b^2\right)\left(\dot{\theta}_b^2 + \sin^2\theta_b \dot{\phi}_b^2\right) &= F_{ex}(t)\dot{r}_b\,, \\ \ddot{\theta}_b + 2\frac{\dot{b}}{b}\dot{\theta}_b + \frac{2}{r_b}\dot{r}_b\dot{\theta}_b - \dot{\phi}_b^2 \sin\theta_b\cos\theta_b &= F_{ex}(t)\dot{\theta}_b\,, \\ \ddot{\phi}_b + 2\frac{\dot{b}}{b}\dot{\phi} + \frac{2}{r_b}\dot{r}_b\dot{\phi}_b + 2\dot{\theta}_b\dot{\phi}_b\cot\theta_b &= F_{ex}(t)\dot{\phi}_b\,, \end{aligned} \tag{4.30}$$

where

$$F_{ex}(t) = a\dot{a}\dot{r}_a^2 + b\dot{b}\left(\frac{\dot{r}_b^2}{1-k_b r_b^2} + r_b^2\dot{\theta}_b^2 + r_b^2\sin^2\theta_b\dot{\phi}_b^2\right). \tag{4.31}$$

Again, $\dot{a} = 0$ will be taken. Due to symmetry, the particle moves on a plane in the extra dimension. Therefore, without loss of generality, $\theta_b = \pi/2$ will be taken and the equations of motion become

$$\begin{aligned} \ddot{r}_a &= \tilde{F}(t)\dot{r}_a\,, \\ \ddot{r}_b + 2\frac{\dot{b}}{b}\dot{r}_b + \frac{k_b r_b \dot{r}_b^2}{1-k_b r_b^2} - r_b\left(1-k_b r_b^2\right)\dot{\phi}_b^2 &= \tilde{F}(t)\dot{r}_b\,, \\ \ddot{\phi}_b + 2\frac{\dot{b}}{b}\dot{\phi} + \frac{2}{r_b}\dot{r}_b\dot{\phi}_b &= \tilde{F}(t)\dot{\phi}_b\,, \end{aligned} \tag{4.32}$$

with

$$\tilde{F}(t) = b\dot{b}\left(\frac{\dot{r}_b^2}{1-k_b r_b^2} + r_b^2\dot{\phi}_b^2\right). \tag{4.33}$$

If the extra dimension is only slowly evoloving, then this set of equations can be solved in powers of $\dot{b}$ again in some short time limit. Thus, in order to evaluate $\tilde{F}(t)$

to the first order of $\dot{b}$, only the leading order of the terms inside bracket is needed. Indeed, in the limit $\dot{b}=0$, the term

$$b^2\left(\frac{\dot{r}_b^2}{1-k_b r_b^2}+r_b^2\dot{\phi}_b^2\right) \tag{4.34}$$

is a constant of motion. Denoting this by ε, the equation governing the motion in the ordinary dimension is given by

$$\ddot{r}_a = H_b\varepsilon\dot{r}_a + O(H_b^2)\,, \tag{4.35}$$

where $H_b \equiv \dot{b}/b$. This is completely analogous to the case where only one extra dimension is needed. This implies that although the particle may have complicated trajectories in the extra dimensional space (notice that there is a point of singularity $r_b = 1$ in the metric), this does not introduce any observable effect in the ordinary dimension other than the exponential growth or decay (depending on the sign of H_b) of the velocity. It can also be concluded from Eq. (4.35) that the short time limit is valid provided that $t^{-1} \gg H_b\varepsilon$.

4.4 Summary

The effect of the extra force on 4*D* particle trajectories under the generalized Robertson-Walker metric has been studied. We have assumed that a 4*D* observer can see the 5*D* particle regardless of the particle's position in the fifth dimension. The Robertson-Walker metric, and its generalized version (Eq. (4.1)) has a very important property that it can always be made conformally flat by appropriate coordinate tranformations, and the extra force depends only on the transformed coordinates. Consequently we found that the extra force is constant in time (to the first order of $\dot{b}$, the rate of the change of the size of the extra dimension), and the deviations of the 4*D* trajectories are in general linear with time. The periodicity of the compactified extra dimensions does not show up in the deviation. The resultant spectrum of the 4*D* projected trajectories obtained by randomizing the initial conditions of the extra coordinates should be the one which is really observed in the laboratory.

In order to make connection with the physical situations, we made use of classical wave packets and compared their evolution to those in the ordinary $4D$ spacetime. It is shown that beside the t^2-dependent term of the usual classical wave packet in $4D$, the evolution of a wave packet in $5D$ also contains t and t^3 terms, which provide a basis of the experimental determinations of the model parameters.

In the next chapter, we want to employ some metric such that the extra dimension is inhomogeneous. We will try to seek some trajectory fluctuations due to the extra dimensional motion in such kind of higher dimensional world.

Chapter 5

Extra Forces in the Randall-Sundrum Model

The generalized Robertson-Walker metric (2.41) has been used in Chapter 6. Because of the homogeneity and isotropy of the extra coordinates (guaranteed by conformal transformations), the extra force is constant in time (to the first order of $\dot{b}$, the rate of the change of the size of the extra dimension), and the deviations of the 4*D* trajectories are in general linear with time. The periodicity of the compactified extra dimensions does not show up in the deviation. If we are to simulate some fluctuations in our models, it may be preferable to introduce some periodic deviations. Therefore, we want to work with some metrics describing inhomogeneous extra dimensions. The Randall-Sundrum metric is one of them because the fifth dimension is warped.

In this chapter, we will examine the effects of the extra force in the Randall-Sundrum model on the 4*D* trajectories. As in Chapter 6, we need first to solve for the trajectories in the Randall-Sundrum space-time so as to examine the behavior of the extra force in this model. In Section 2.5, the setup of the RSI scenario has been reviewed and the corresponding metric has been obtained. As we have seen in Section 3.5, no extra force can be detected when the dynamical variables are expressed in terms of the 4*D* proper time or the coordinate time. Therefore, we will slightly modify the metric such that a coupling between the extra dimension and the ordinary dimensions become possible. Notice that we will also assume that an $4D$ observer can see the $5D$ particle regardless of its position in the fifth dimension. In Section 5.2, the particle trajectories are solved in some short time limits. Then we will focus on the trajectory fluctuations embedded in the $4D$ trajectories due to the unobserved extra dimensional motions in Section 5.3, and we will show how the evolution of a free wave packet is modified. Finally, in Section 5.4, we have a short discussion on how the periodic fluctuation depends on the inhomogeneity of the warped extra dimension.

5.1 Model Development - Evolving Extra Dimension

In order to introduce extra forces in the model, a coupling between the extra dimension and the ordinary spacetime can be made by a simple generalization of the metric (2.48). The experience in dealing with the generalized Robertson-Walker metric suggests that a time dependence of the size of the extra dimension can be introduced in (2.48):

$$d\hat{S}^2 = e^{-2kb(t)|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu - b^2(t)d\phi^2. \tag{5.1}$$

The payoff of this modification is that the classical action holds only if it satisfies some adiabatic conditions. Indeed, the Einstein equations derived from the metric (5.1) are given by

$$\begin{gathered}\sigma'^2 - \frac{\sigma e^{2\sigma}}{2}\left(\sigma\dot{b}^2 - \frac{\ddot{b}}{k}\right) = -\frac{\Lambda}{24M^3k^2},\\ \dot{b}^2 e^{2kb\sigma}(kb-\sigma) + \sigma'' = \frac{V_{\text{vis}}}{12M^3k}\delta(\phi-\pi) + \frac{V_{\text{hid}}}{12M^3k}\delta(\pi).\end{gathered} \tag{5.2}$$

To go further, two assumptions will be made: (i) $-\sigma e^{2\sigma}\left|\sigma\dot{b}^2 - \ddot{b}/k\right| \ll -\Lambda/(24M^3k^2)$ and (ii) the contribution of the singularity comes from the second derivative of $\sigma(\phi)$. Therefore the stationary solution Eq. (2.48) holds, and the adiabatic criteria are

$$e^{2kb|\phi|}\left|\frac{\ddot{b}}{k}|\phi| - \phi^2\dot{b}^2\right| \ll 1 \quad \text{and} \quad \dot{b}^2 e^{2kb|\phi|}\Big|kb - |\phi|\Big| \ll 1. \tag{5.3}$$

These two conditions must be satisfied simultaneously throughout the evolution.

Particle trajectories in the Randall-Sundrum braneworld will be examined. Assuming one ordinary dimension x, the equations of motion are given by

$$\frac{d^2t}{d\hat{S}^2} - k\dot{b}|\phi|\left[\left(\frac{dt}{d\hat{S}}\right)^2 + \left(\frac{dx}{d\hat{S}}\right)^2\right] + e^{2kb|\phi|}b\dot{b}\left(\frac{d\phi}{d\hat{S}}\right)^2 - 2kbs_\phi\frac{dt}{d\hat{S}}\frac{d\phi}{d\hat{S}} = 0,$$

$$\frac{d^2x}{d\hat{S}^2} - 2k\dot{b}|\phi|\frac{dt}{d\hat{S}}\frac{dx}{d\hat{S}} - 2kbs_\phi\frac{dx}{d\hat{S}}\frac{d\phi}{d\hat{S}} = 0,$$

$$\frac{d^2\phi}{d\hat{S}^2} + 2\frac{\dot{b}}{b}\frac{dt}{d\hat{S}}\frac{d\phi}{d\hat{S}} + \frac{k}{b}s_\phi e^{2kb|\phi|}\left[\left(\frac{dx}{d\hat{S}}\right)^2 - \left(\frac{dt}{d\hat{S}}\right)^2\right] = 0,$$

where s_ϕ is the sign function

$$s_\phi = \begin{cases} 1, & \text{if } \phi > 0; \\ -1, & \text{if } \phi < 0. \end{cases}$$

In terms of the coordinate time t,

$$\ddot{x} = k\dot{b}|\phi|(1-\dot{x}^2)\dot{x} + b\dot{b}e^{2kb|\phi|}\dot{\phi}^2\dot{x}, \tag{5.4}$$

$$\ddot{\phi} + 2kbs_\phi\dot{\phi}^2 - \frac{k}{b}s_\phi e^{-2kb|\phi|}(1-\dot{x}^2) = -2\frac{\dot{b}}{b}\dot{\phi} - k\dot{b}|\phi|(1+\dot{x}^2)\dot{\phi} + b\dot{b}e^{2kb|\phi|}\dot{\phi}^3. \tag{5.5}$$

The energy E of the particle is an adiabatic constant of motion

$$E = \frac{e^{-kb|\phi|}M_5c^2}{\sqrt{1-(\dot{x}^2 + e^{2kb|\phi|}b^2\dot{\phi}^2)/c^2}}, \tag{5.6}$$

where M_5 is the $5D$ mass. The physically measurable velocity in the extra dimension is given by $v_\phi \equiv e^{kb|\phi|}b\dot{\phi}$.

The coupled set of Eq. (5.4) and Eq. (5.5) is highly non-linear, and it is difficult to find its exact solution in general. Thus it will be solved by successive approximation in the first power of the derivative $\dot{b}(t)$ under some appropriate short time limits. For sufficiently slow varying $b(t)$, the energy E is a quasi-constant of motion which proves to be very useful in determining the motions of the particle.

Our goal is to find the approximate solution for $x(t)$ up to the first power of $\dot{b}(t)$. It suffices to solve for $\phi(t)$ by simply setting $\dot{b} = 0$ in Eq. (5.5) and taking $b(t)$ and

Figure 5.1: The phase plot for $\phi(t)$, with $k = 20$, $b_0 = 0.01$ and $\dot{x}_0 = 0.01c$. The vector field is defined as $N^{-1}(\dot{\phi}, \dot{v}_\phi)$, N being the normalization constant. One phase path (narrow) passes through the point $(-\pi, 0.9c)$ and the other phase path (thick) passes through the point $(-\pi, 0.6c)$.

$\dot{x}(t)$ as the initial values b_0 and $\dot{x}_0$ respectively. Eq. (5.5) can then be written as

$$be^{kb|\phi|}\frac{dv_\phi}{dt} = s_\phi kb(1 - \dot{x}^2 - v_\phi^2). \tag{5.5'}$$

A phase plot described by Eq. (5.5') is shown in Fig. 5.1. The vector field is defined as $N^{-1}(\dot{\phi}, \dot{v}_\phi)$, N being the normalization constant. An attractor is located at $(\pm\pi, 0)$. Only two kinds of motions are possible: in the low energy regime (to be defined in the following paragraph), the particle oscillates around $(\pm\pi, 0)$; in the high energy regime, on the other hand, the particle could run through the whole fifth dimension unidirectionally. A particular example for each regime is shown in Fig. 5.1.

The critical condition that separates the oscillatory and monotonic regimes of the motions could be found from energy consideration. Right at the critical energy, the particle becomes motionless at the hidden brane, *i.e.* $\dot{\phi} = 0$ at $\phi = 0$. Thus, in the limit of $\dot{b} = 0$, the critical energy is given by

$$E_{cr} = \frac{e^{-kb_0|\phi_0|} M_5 c^2}{\sqrt{1-\dot{x}_0^2}}. \tag{5.7}$$

An energy slightly smaller than E_{cr} would result in an oscillatory motion and otherwise a monotonic motion.

These conclusions are consistent with what we found in the quantum mechanical calculations (Section 2.5). It has been found that the Kaluza-Klein modes are peaked near the visible brane, and thus the Kaluza-Klein excitations are most likely to be found there. By the correspondence principle [45], it is expected that the classical particle should spend most of its time in the region near the visible brane.

5.2 Classical Particle Trajectories

Eq. (5.5') can be solved for $\phi(t)$ in the intervals $[-\pi, 0]$ and $[0, \pi]$, which are disconnected in the phase space:

$$\phi(t) = \frac{s_{\phi_0}}{2kb_0} \ln g\big(k(t-t_0)\big), \tag{5.8}$$

where

$$g(\zeta) \equiv (1-\dot{x}_0^2)\zeta^2 + e^{kb_0|\phi_0|}\Big(e^{kb_0|\phi_0|} + 2s_{\phi_0} v_{\phi_0} \zeta\Big) \tag{5.9}$$

for some reference time t_0, and the suffix 0 denotes initial values at the moment t_0. The initial values must be modified each time when the particle enters the intervals from the other in the extra dimension. The corresponding $\dot{x}(t)$ in the first power of $\dot{b}_0$ is given by

$$\dot{x}(t) = \frac{\dot{x}_0\Big\{1 + \frac{\dot{b}_0}{4kb_0}\dot{g}\big(k(t-t_0)\big)\ln g\big(k(t-t_0)\big)\Big\}}{1 + b_0\dot{b}_0\phi_0\dot{\phi}_0 e^{2kb_0|\phi_0|}}. \tag{5.10}$$

The initial conditions must also be modified accordingly when ϕ enters into the two intervals. This expression is accurate provided the whole factor multiplying $\dot{x}_0$ is close to unity. Thus this defines the short-time limit. The trajectories $x(t)$ in the ordinary dimension can be obtained directly from the integration of Eq. (5.10).

We would like to extend the solution Eqs. (5.8) and (5.10) to the whole time domain (within the short time limit). Let the time of reference be $t_0 \equiv 0$. Two cases will be considered: (i) For $E > E_0$, The latest moment t_{vis} when the particle is at the visible brane can be solved from the root of the following equation:

$$t_{\text{vis}} = \frac{\sqrt{(1-\dot{x}_0^2)\left[e^{2kb_0(|\pi|-|\phi_0|)}-1\right]+v_{\phi_0}^2}-|v_{\phi_0}|}{s_{\phi_0}s_{\dot{\phi}_0}e^{-kb_0|\phi_0|}k(1-\dot{x}_0^2)}. \tag{5.11}$$

Similarly, the latest moment t_{hid} when the particle is at the hidden brane can be solved from

$$t_{\text{hid}} = \frac{\sqrt{(1-\dot{x}_0^2)\left[e^{-2kb_0|\phi_0|}-1\right]+v_{\phi_0}^2}-|v_{\phi_0}|}{s_{\phi_0}s_{\dot{\phi}_0}e^{-kb_0|\phi_0|}k(1-\dot{x}_0^2)}. \tag{5.12}$$

Either of these quantities is negative and the other positive. The period of motion for $\phi(t)$ is clearly $\tau_\phi \equiv 2|t_{\text{vis}} - t_{\text{hid}}|$. (ii) For $E < E_0$, the turning point is reached at the time

$$t_{\text{hid}} = -s_{\phi_0}s_{\dot{\phi}_0}\frac{e^{kb_0|\phi_0|}|v_{\phi_0}|}{k(1-\dot{x}_0^2)}, \tag{5.13}$$

where the time is still suffixed by “hidden” for convenience. The period in this case is double of that in the previous case, *i.e.*

$$\tau_\phi = 4|t_{\text{vis}} - t_{\text{hid}}| = \frac{\sqrt{(1-\dot{x}_0^2)\left[e^{2kb_0(|\pi|-|\phi_0|)}-1\right]+v_{\phi_0}^2}}{e^{-kb_0|\phi_0|}k(1-\dot{x}_0^2)/4}. \tag{5.14}$$

$\phi(t)$ in the limit of $\dot{b} = 0$ can now be written in terms of the coordinate-time as

$$\phi(t) = s_{\phi_0}\frac{\dot{\Delta}(t)}{2kb_0}\ln g\big(k\Delta(t)\big), \tag{5.15}$$

where $\Delta(t)$ is a triangular wave function defined by

$$\Delta(t) = \left|\Big((t-t_a)\text{ Mod }\tau_\phi\Big) - \frac{\tau_\phi}{2}\right| + t_b,$$

where $t_a \in \text{Max}\{t_{\text{hid}}, t_{\text{vis}}\} > 0$ is the shortest time after a reference time $t = 0$ when the particle arrives at a brane (either visible or hidden) or the turning point, and $t_b = \text{Min}\{t_{\text{hid}}, t_{\text{vis}}\} < 0$ is the latest time before $t = 0$ when the particle passes through the other brane or the turning point. The value of the time derivative $\dot{\Delta}(t)$ is either $+1$ or -1, with the initial value chosen always to be $+1$.

$\dot{x}(t)$ for all time t correct to the first order in $\dot{b}$ is given by

$$\dot{x}(t) = \frac{\dot{x}_0\left\{1+\dot{b}_0\left[\alpha\, n(t)+2kh'\big(k\Delta(t)\big)\right]\right\}}{1+b_0\dot{b}_0\phi_0\dot{\phi}_0 e^{2kb_0|\phi_0|}}, \tag{5.16}$$

where

$$h(\zeta) = \frac{1}{4k^2b_0}\left\{g(\zeta)\big(\ln g(\zeta)-1\big)-e^{2kb_0|\phi_0|}\big(2kb_0|\phi_0|-1\big)\right\}$$

and

$$\alpha = 2k\left|h'(kt_a^-)-h'(kt_b^+)\right|.$$

$n(t)$ is the floor function (i.e. the greatest integer of the given independent variable) defined by

$$n(t) = \left\lfloor \frac{t-t_{\text{vis}}}{\tau_x}\right\rfloor + \theta(t_{\text{vis}}),$$

where $\theta(x)$ is the Heaviside step function

$$\theta(x) = \begin{cases} 0, & x<0\,; \\ 1, & \text{otherwise.}\end{cases}$$

Due to the $S_1/\mathbb{Z}_2$ symmetry, the period τ_x observed in $\dot{x}(t)$ is

$$\tau_x = 2|t_{\text{vis}}-t_{\text{hid}}| \tag{5.17}$$

irrespective of the energy E. Since $h'(kt_{\text{hid}})=0$ for both energy regimes, one gets

$$\alpha = 2\pi e^{kb_0|\phi_0|}\sqrt{(1-\dot{x}_0^2)[e^{2kb_0(\pi-|\phi_0|)}-1]+v_{\phi_0}^2}. \tag{5.18}$$

The short time limit can be written explicitly as

$$t \ll \frac{\tau_x}{\dot{b}_0\alpha}. \tag{5.19}$$

Another integration of $\dot{x}(t)$ gives

$$x(t) = x_0+\frac{\dot{x}_0\left\{t+\dot{b}\left[\alpha\tau_x n(t)\left(\frac{t-t_{\text{vis}}}{\tau}-\frac{n(t)-1}{2}\right)+h\big(k\Delta(t)\big)\right]\right\}}{1+b\dot{b}\phi_0\dot{\phi}_0e^{2kb|\phi_0|}}. \tag{5.20}$$

Two examples of the trajectories are shown in Fig. 5.2. These trajectories are constructed based on the phase paths in ϕ highlighted in Fig. 5.1.

Figure 5.2: Two examples of particle trajectories in an Randall-Sundrum world corresponding to the phase paths in Fig. 5.1. Narrow: $v_{\phi_0} = 0.9c$. Thick: $v_{\phi_0} = 0.6c$. Both start at $\phi_0 = -\pi$ and $\dot{x}_0 = 0.01c$.

Discussion

The effect of the extra force on $\dot{x}(t)$ can be understood as follows. From Eq. 5.4, the extra force is given by

$$F_{\text{ex}} = \frac{\dot{b}}{b}\left[kb|\phi|(1-\dot{x}^2)+v_\phi^2\right]\dot{x}. \tag{5.21}$$

This extra force is parallel to its ordinary velocity $\dot{x}$. As in Chapter 6, the extra force seems to be dependent on the choice of reference frames (where $\dot{x}$ may be different) [31]. In the approximation up to first order of $\dot{b}$, F_{ex} can be evaluated by using Eq. (5.8) and by taking $\dot{x}$ as a constant since the ordinary dimensions are flat. The quantity inside the square bracket is always non-negative. That means, whether the extra force causes acceleration or deceleration on the particle depends only on the sign of $\dot{b}$. Suppose $\dot{b} > 0$. When $|\phi|$ decreases from π, the phase plot (Fig. 5.1) suggests that v_ϕ^2 always decreases, and so does the quantity inside the whole square bracket. Conversely, if ϕ grows towards π, the quantity inside the square bracket increases. Thus, the rate of increase in $\dot{x}(t)$ is greatest at $\phi = \pi$ while it is smallest at $\phi = 0$ or at the turning points in the extra dimension. In other words, the extra force behaves like a sinusoidal periodic force, and $\dot{x}(t)$ evolves in a way like the step function if $\dot{x}(0) \neq 0$. Similar conclusion can be obtained directly if $\dot{b} < 0$, but in this case $\dot{x}$ decreases with time like the step function. Obviously, the step-function behavior of $\dot{x}$ implies that we can decompose $\dot{x}$ into a linear trend and a see-saw like, periodic function. We have already seen the linear trend in Chapter 6. This linear trend is due to the evolution of the size of the extra dimension. The see-saw like, periodic function is something new. This is clearly because of the inhomogeneity of the extra dimension, in which the velocity v_ϕ varies with the extra coordinate ϕ so as to keep the term $(e^{-kb|\phi|}/\sqrt{(1-\dot{x}^2-v_\phi^2)})$ constant (since the total energy is constant up to first order of $\dot{b}$). Similar arguments suggest that there is also a second order trend plus a see-saw like, periodic function in $x(t)$.

Thus, in the first order of $\dot{b}$, $\dot{x}(t)$ can be decomposd into an initial value $\dot{x}_0$, a linear trend and a very small oscillatory component:

$$\dot{x}(t) = x_0 + \dot{x}_\ell(t) + \dot{x}_{\rm osc}(t) \tag{5.22}$$

where the linear dependence is given by

$$\dot{x}_\ell(t) = \dot{b}_0 \alpha \dot{x}_0 t/\tau_x\,, \tag{5.23}$$

and the see-saw like oscillatory component is given by

$$\dot{x}_{\rm osc}(t) = \dot{b}_0 \dot{x}_0 \left\{\alpha\xi(t) + 2kh'\big(k\Delta(t)\big)\right\}, \tag{5.24}$$

with $\xi(t) \equiv n(t) - t/\tau_x$ being a see-saw function with period τ_x, which is also the period of $\Delta(t)$. Eq. (5.23) and Eq. (5.24) both depend on the initial conditions ϕ_0 and $\dot{\phi}_0$, and the parameters k and b_0. As time evolves, it is obvious that the linear part becomes dominant. The average value of the acceleration $\ddot{x}(t)$ can be characterized by the slope of $x_\ell(t)$, *i.e.* the dimensionless quantity $b_0\alpha/\tau_x$, which is a function of kb_0 and its values are shown in Fig. 5.3. It is remarked that when $E < E_0$, (*i.e.* when the particle is unable to pass through the hidden brane), $b_0\alpha/\tau_x$ has the constant value $kb_0\pi(1-\dot{x}_0^2)$. It is clear from Fig. 5.3 that for large value of kb_0, $b_0\alpha/\tau_x$ in most of the parameter space (ϕ_0, v_{ϕ_0}) takes up this constant value. Thus we will see in the next section that a particle moving freely in a five dimensional Randall-Sundrum world (as observed by a 4*D* observer) always experiences a constant extra force $\pi k\dot{b}_0\dot{x}_0(1-\dot{x}_0^2)/c$. This extra force depends on the initial velocity of the particle and is thus very similar to the magnetic force on a charged particle with a cyclotron frequency $qB/m = \pi k\dot{b}_0(1-\dot{x}_0^2)/c$. The oscillatory component would then present some sort of fluctuation on the observed trajectory. Such fluctuations appear completely random for repeated experiments with identical settings in the ordinary dimension since the initial conditions ϕ_0 and $\dot{\phi}_0$ cannot be determined in the laboratory. In the same token, $x(t)$ can be decomposed into four parts:

$$x(t) = x_0 + \dot{x}_0 t + x_q(t) + x_{\rm osc}(t), \tag{5.25}$$

Figure 5.3: The values of $b_0\alpha/\tau_x$ in the parameter space ϕ_0, v_{ϕ_0}. 1000 sample points were chosen by randomizing ϕ_0 and v_{ϕ_0} in the intervals $[-\pi,\pi]$ and $[-c,c]$ respectively. The contours are created after triangulation of the sample points. Filled area represents the parameter space where $E < E_0$. Lower: $kb_0 = 4$; upper: $kb_0 = 0.2$.

where

$$x_q(t) = \dot{x}_0 \dot{b}_0 \frac{\alpha}{\tau_x} \frac{t(t-2t_{\text{hid}})}{2} - \dot{x}_0 b_0 \dot{b}_0 \phi_0 \dot{\phi}_0 e^{2kb_0|\phi_0|} t, \tag{5.26}$$

and the see-saw like oscillatory component

$$x_{\text{osc}}(t) = \dot{b}_0 \dot{x}_0 \left\{ \alpha \tau_x \xi(t) \left[\frac{t-t_{\text{vis}}}{\tau_x} - \frac{n(t)+t-1}{2} \right] + h\big(\Delta(t)\big) \right\}. \tag{5.27}$$

Again, for $E < E_0$, the coefficient of the quadratic term in $x_q(t)$ is constant at $\dot{b}_0 \alpha / \tau_x = k\dot{b}_0(1-\dot{x}_0)$.

The most interesting case occurs when $\dot{b} > 0$. In this case, the phase line $\dot{x} = 0$ is unstable, and we have to consider the evolution of the density functions in the phase space [50]. As in the previous chapter, we will assume that all initial conditions have distributions around some mean values. The standard deviations or uncertainties of these initial conditions form phase space volumes, and the evolution of these volumes will resemble the evolution of the wave packets in quantum mechanics.

5.3 Unobserved Motions in Extra Dimensions

Assume that a 4*D* observer can see the 5*D* particle, regardless of its position in the extra dimension. This is different from the basic assumption in brane theories that a 5*D* particle can only be seen when it is confined in the visible brane. As in the last chapter, we will randomize the initial conditions x_0, $\dot{x}_0$, ϕ_0 and $\dot{\phi}_0$, and we will obtain a formula governing the spread of the wave packets of a free particle. Eq. (5.16) and Eq. (5.20) show that if the initial conditions ϕ_0 and $\dot{\phi}_0$ are randomized, $x(t)$ and $\dot{x}(t)$ would show deviations from the usual 4*D* trajectories, and they must be described in some probability distributions. Before proceeding to the calculations, we need to determine the distribution functions for the extra coordinates.

Again, consider the 4*D* rest mass, which is related to the 5*D* rest mass in the Randall-Sundrum model by

$$m_0 = \frac{e^{-kr_b|\phi|} M_5}{\sqrt{1 - r_b^2 e^{2kr_b|\phi|} (\frac{d\phi}{ds})^2}}\,, \tag{5.28}$$

where M_5 is the 5D rest mass. In the non-relativistic limit $x(t) \ll 1$, whence $ds \approx dt$, m_0 is equivalent to the total energy E in Eq. (5.6). Thus, m_0 is a constant and is determined from the initial conditions of the extra coordinates. Again, in order not to contradict to our observations about the 4D rest mass, we require that ϕ_0 and $\dot{\phi}_0$ must be peaked around some definite values and with very narrow width of the distributions. From the trajectories solved in the last section, we know that $\phi = \pi$ is a stable point in the phase space, and thus the particle can stay in the proximity around $\phi = \pi$. On the contrary, there is no stable point around $\phi = 0$, and the particle cannot stay around $\phi = 0$. Thus we may assume that ϕ_0 is gaussianly peaked at π with an uncertainty (standard deviation) $\Delta\phi_0 \ll \pi$:

$$P_{\phi_0}(\phi_0) = \frac{1}{\sqrt{2\pi\Delta\phi_0^2}} \exp\left\{-\frac{\phi_0^2}{2\Delta\phi_0^2}\right\}. \qquad (\Delta\phi_0 \ll \pi) \tag{5.29}$$

Furthermore, in the large kb_0 limit, the particle will mostly oscillate around $\phi = \pi$ unless it has enough energy to pass through the hidden brane. Therefore, it is natural to assume that $\dot{\phi}_0$ is gaussianly peaked at zero with an uncertainty $\Delta\dot{\phi}_0 \ll 1$, which has the same form as given by Eq. (4.12):

$$P_{v_{\phi 0}}(v_{\phi 0}) = \frac{1}{\sqrt{2\pi\Delta v_{\phi 0}^2}} \exp\left\{-\frac{v_{\phi 0}^2}{2\Delta v_{\phi 0}^2}\right\}. \qquad (\Delta v_{\phi 0} \ll 1) \tag{5.30}$$

Essentially, we are considering only cases where the particle performs oscillatory motion around the visible brane: $E < E_0$.

Under these probability distributions, the mean value of the 4D rest mass is

$$\begin{aligned}
\overline{m}_0 &= \int P_{\phi_0} d\phi_0 P_{v_{\phi 0}} dv_{\phi_0} \frac{e^{-kr_b|\phi_0|} M_5}{\sqrt{1 - v_{\phi 0}^2}} \\
&\doteq e^{\frac{(kb_0\Delta\phi_0)^2}{2}} \left(1 - \frac{\Delta v_{\phi 0}^2}{2}\right) \mathrm{Erfc}\left(\frac{kb_0\Delta\phi_0}{\sqrt{2}}\right) \\
&\doteq M_5 \left(1 - \frac{kb_0\Delta\phi_0}{\sqrt{\pi/2}}\right),
\end{aligned} \tag{5.31}$$

where

$$\mathrm{Erfc}(z) = 1 - \frac{2}{\pi} \int_0^z e^{\frac{y^2}{2}} dy, \tag{5.32}$$

and the approximations are made by the assumptions that $\Delta\phi_0$, $\Delta v_{\phi 0} \ll 1$. All terms of orders higher than $\Delta\phi_0$ and $\Delta v_{\phi 0}$ are ignored. Similarly, the uncertainty of the 4D rest mass around the mean $\overline{m}_0$ is given by

$$\begin{aligned}\Delta m_0^2 &= \int P_{\phi_0} d\phi_0 P_{v_{\phi 0}} dv_{\phi_0} \left(\frac{e^{-kr_b|\phi_0|} M_5}{\sqrt{1-v_{\phi 0}^2}} - \overline{m}_0^2 \right)^2 \\ &\doteq M_5^2 k^2 b_0^2 \Delta\phi_0^2 \left(1 - \frac{2}{\pi}\right) \\ &\approx \overline{m}_0^2 k^2 b_0^2 \Delta\phi_0^2 \left(1 - \frac{2}{\pi}\right), \end{aligned} \tag{5.33}$$

where the last approximation is made up to $\Delta\phi_0^2$. Thus the 4D rest mass is most sensitive to the distribution of ϕ_0.

In the following, we develop a parallel treatment as in Chapter 6. We first consider the case where only ϕ_0 and $\dot{\phi}_0$ are randomized, and see how the final position and the final velocity in the ordinary dimensions distribute.

Case 1: $\Delta x_0 = 0$, $\Delta\dot{x}_0 = 0$, $\phi_0 \neq 0$ and $\dot{\phi}_0 \neq 0$.

Since the probability of ϕ_0 and $\dot{\phi}_0$ ensures the total energy being lower than the critical energy Eq. (5.7), the linear trend of $\dot{x}(t)$ and the quadratic trend of $x(t)$ are fixed, and only the oscillatory components vary with ϕ_0 and $\dot{\phi}_0$. These oscillatory components are responsible for the uncertainties in the measurements if the system is allowed to run for a time much longer than the period of oscillation τ_x. In the large kb_0 limit such that $e^{kb_0\pi} \gg 1$, the period of x_{osc}, $\langle\tau_x\rangle_{\phi,v_\phi}$, is of the order $e^{kb_0\pi}/kc$ if $\dot{x}_0 \ll 1$. For example, if $kb_0 \sim 10$ and k is of the order of the Planck scale, $\langle\tau_x\rangle_{\phi,v_\phi} \sim 10^{-31}$ s. Thus the period is too short that these oscillatory components appear completely random in nature and the occurrence of their possible values can only be described by probability distribution function. We define the expectation value of a physical quantity F by

$$\langle F(t)\rangle_{\phi,v_\phi} = \int P_{\phi_0} d\phi_0 P_{v_{\phi 0}} dv_{\phi 0} \left[F(t) - F_{\text{osc}}(t)\right] \quad (t \gg \tau_x) \tag{5.34}$$

after the randomization of ϕ_0 and $\dot{\phi}_0$, where P_{ϕ_0} and $P_{v_{\phi 0}}$ are given by Eq. (5.29) and Eq. (5.30) respectively. The oscillatory component of F is eliminated from the expectation value. Since the oscillatory components are to be taken into account as the source of experimental fluctuation, the uncertainty ΔF is defined as

$$\Delta F^2 = \lim_{\zeta\to\infty}\int_{\zeta}^{\zeta+\tau_x} dt \int P_{\phi_0} d\phi_0 P_{v_{\phi 0}} dv_{\phi 0} \Big(F(t) - \langle F(t)\rangle_{\phi,v_\phi}\Big)^2, \tag{5.35}$$

so that the uncertainty is of the order of the amplitude of $F_{\text{osc}}(t)$.

In the non-relativistic limit $\dot{x}_0 \ll 1$, the mean value of $x(t)$ is given by

$$\langle x(t)\rangle_{\phi,v_\phi} \doteq x_0 + \dot{x}_0 t(1 + \frac{\pi k\dot{b}_0}{2}t), \tag{5.36}$$

where we have made used of the fact that $\langle b_0\alpha/\tau_x\rangle_{\phi,v_\phi} = kb_0\pi$ and $\langle t_{\text{hid}}\rangle_{\phi,v_\phi} = 0$. The equality is made with the short time limit, $t^{-1} \gg \pi k\dot{b}_0$. The uncertainty Δx comes from the oscillatory component x_{osc} and it is approximately equal to the amplitude of $x_{\text{osc}}(t)$:

$$\Delta x \doteq \dot{b}_0\dot{x}_0\alpha\tau_x = k\dot{b}_0\pi\dot{x}_0\tau_x^2 \approx \pi\dot{b}_0\dot{x}_0 e^{2kb\pi}/kc^2.$$

It is important to notice that in the short time limit, the uncertainty is constant in time. An example of the probability distributions of $x(t)$ is shown in the lower figure of Fig. 5.4.

The momentum p_x is defined as $m_0\dot{x}$, where m_0 is the 4D rest mass. Its expectation value is just the product of $\langle m_0\rangle_{\phi,v_\phi} = \overline{m}_0$ and $\langle x(t)\rangle_{\phi,v_\phi}$:

$$\langle p_x(t)\rangle \doteq \overline{m}_0\dot{x}_0\Big(1 + k\dot{b}_0\pi t\Big). \tag{5.37}$$

The uncertainty Δp_x is contributed from both Δm_0 and $\Delta\dot{x}$, where $\Delta\dot{x} \approx \dot{b}_0\dot{x}_0\alpha \approx \pi\dot{b}_0\dot{x}_0 e^{kb_0\pi}/c$:

$$\Delta p_x^2 = \dot{x}(t)^2\Delta m_0^2 + \overline{m}_0^2\Delta\dot{x}^2 = \dot{x}_0^2\Delta m_0^2 + \overline{m}_0^2\Delta\dot{x}^2 + 2\pi k\dot{b}_0\Delta m_0^2\dot{x}_0^2 t.$$

which has a linear time dependence in the short time limit.

Figure 5.4: The probability of getting a particular value of (a) $\dot{x}_{\rm osc}$ and (b) $x_{\rm osc}$ in Case 1. The result is simulated by superimposing 2000 trajectories after an evolution for a time much greater than the period of $\langle \tau_x \rangle_{\phi, v_\phi}$ with randomly chosen ϕ_0 $\dot{\phi}_0$. 100 bins are used in the range shown in the figure. The area under the curve is normalized to 1. $b_0 = 10^{-35}$ m, $\dot{b}_0/cb_0 = 10^{-10}$ m^{-1}, $kb_0 = 10$ and $\dot{x}_0 = 0.01c$. The most probable values of the fluctuations are found near the maximum values, where the rates of change of $x_{\rm osc}$ become zero. The widths of the distributions are given by $\Delta\dot{x}$ and Δx respectively.

Case 2: Δx_0**,** $\Delta \dot{x}_0$**,** ϕ_0 **and** $\dot{\phi}_0 \neq 0$

Since $\dot{x} = 0$ is an unstable phase line in the phase space when $\dot{b} > 0$, it is more appropriate to consider the evolution of the phase space volumes around this unstable phase line. We assume that Δx_0 and $\Delta \dot{x}_0 \neq 0$, and we randomize the extra coordinates at the same time. It is straightforward to extend the previous calculations if $\Delta x_0 \neq 0$ and $\Delta \dot{x}_0 \neq 0$. We are interested only in the uncertainties. For $x(t)$,

$$\Delta x(t)^2 \approx \Delta x_0^2 + \Delta \dot{x}_0^2 \left(t^2 + \frac{\pi^2 \dot{b}_0^2 e^{4kb\pi}}{k^2 c^4} \right). \tag{5.38}$$

The second term in the bracket comes from the oscillatory component $x_{\rm osc}(t)$. For $p_x(t)$,

$$\Delta p_x(t)^2 \doteq \Delta p_{x0}^2 \left(1 + 2\pi k \dot{b}_0 t \right). \tag{5.39}$$

where

$$\Delta p_{x0}^2 = \overline{m}_0^2 \Delta \dot{x}_0^2 \left[1 + \frac{\pi^2 \dot{b}_0^2 e^{2kb_0\pi}}{c^2} + k^2 b_0^2 \Delta \phi_0^2 \left(1 - \frac{2}{\pi} \right) \right]. \tag{5.40}$$

The second term in the square bracket of Δp_{x0} are due to the oscillatory component of $\Delta \dot{x}(t)$ and the third term is due to the variation of the $4D$ mass.

Hence the product $\Delta x(t)^2 \Delta p_x(t)^2$ evolves with time as

$$\begin{aligned} \Delta x(t)^2 \Delta p_x(t)^2 =& h_0^2 + \frac{\Delta p_{x0}^4}{\overline{m}_0^2} t^2 + 2\pi k \dot{b}_0 t \left(h_0^2 + \frac{\Delta p_{x0}^4}{\overline{m}_0^2} t^2 \right) \\ &+ \frac{2 \Delta p_{x0}^4}{\overline{m}_0^2} \left[k^2 b_0^2 \Delta \phi_0^2 \left(1 - \frac{2}{\pi} \right) \right] t^2 + \frac{\Delta p_{x0}^4}{\overline{m}_0^2} \left(\frac{\pi \dot{b}_0 e^{2kb\pi}}{kc^2} \right), \end{aligned} \tag{5.41}$$

where

$$h_0^2 \equiv \Delta x_0^2 \Delta p_{x0}^2. \tag{5.42}$$

h_0 is required to be non-zero since we are considering the evolution of the phase space volumes around $\dot{x} = 0$. Eq. (5.42) is analogous to Eq. (4.27). This equation is evaluated up to the first order in $\dot{b}_0$. The third term on the right hand side is a correction term which is proportional to $k\dot{b}_0$. Again, $k\dot{b}_0$ is of cosmological time scale, and thus Eq. (5.42) essentially reproduces a $4D$ free wave packet in the short time limit. The fourth term comes from the mass distribution of m_0 in the definition of p_x, and the fifth term is due to the oscillatory component $x_{\text{osc}}(t)$.

Eq. (5.42) provides a classical test of 5D particle kinematics. Compared to Eq. (4.27), the last three terms on the right hand side are the correction terms of the usual evolution of a free wave packet if $\overline{m}_0$ is taken to be the rest mass of the particle. Since h_0 and Δp_{x0} are measured in the beginning of the experiment, by the spread of the wave packet as variables of the coordinate time t, the two coefficients of three correction terms can be used to solve for $\Delta\phi_0$, b_0 and $\dot{b}_0$, with $kb_0 \sim 10$ in order to solve the Hierarchy Problem.

5.4 Modification of RSI model

From the RSI model, the following observation has been made: the fluctuations in trajectories in the ordinary space-time due to the extra dimensional motions can be realized provided that the extra dimension is inhomogeneously compactified and that it is evolving. This is completely different from what is obtained from the generalized Robertson-Walker metric, in which no such fluctuations are found because the extra dimension is conformally flat. Therefore the periodicity of the fluctuations is due to the inhomogeneity of the extra dimension. It is desirable to examine the effect of inhomogeneity in more detail.

In this section, a modified version of the Randall-Sundrum metric with adjustable inhomogeneity of the fifth dimension will be employed:

$$d\hat{S}^2 = e^{-2kb(t)\cos n\phi}(dt^2 - dx^2) - b^2(t)d\phi^2, \tag{5.43}$$

where n serves as a measure of the inhomogeneity of the extra dimension. The equations of motion in the time coordinate are given by

$$\ddot{x} = e^{2kb(t)\cos n\phi} b\dot{b}\dot{\phi}^2 \dot{x} + k\dot{b}\left(1-\dot{x}^2\right)\dot{x}\cos n\phi, \tag{5.44}$$

$$\ddot{\phi} + 2\frac{\dot{b}}{b}\dot{\phi} = -\frac{nk}{b}e^{-2kb\cos n\phi}\left(1-\dot{x}^2\right)\sin n\phi + 2nkb\dot{\phi}^2\sin n\phi \tag{5.45}$$
$$+ e^{2kb\cos n\phi} b\dot{b}\dot{\phi}^3 - k\dot{b}\left(1+\dot{x}^2\right)\dot{\phi}\cos n\phi,$$

and the energy in such a scenario is defined by

$$E = \frac{e^{-kb\cos n\phi}}{\sqrt{1-\dot{x}^2 - e^{2kb\cos n\phi} b^2 \dot{\phi}^2}}. \tag{5.46}$$

To calculate $x(t)$ up to the first order in $\dot{b}$, only the leading term of $\phi(t)$ is needed. That is, $\phi(t)$ will be solved by setting $\dot{b} = 0$. The general features of $\phi(t)$ can be examined from its phase plot. From the equation of motion,

$$be^{kb\cos n\phi}\frac{dv_\phi}{dt} = -nkb(1-\dot{x}^2 - v_\phi^2)\sin n\phi \quad \text{and} \quad |v_\phi \equiv be^{kb\cos n\phi}\dot{\phi}| < c,$$

which has the same form as that of the pendulum under gravity. Fig. 5.5 shows an example of the phase plots of v_ϕ versus ϕ. The vector field is defined as $N^{-1}\left(\dot{\phi}, \dot{v}_\phi\right)$, where N is the normalization constant. Generally speaking, attractors are located at $\phi = \pm j2\pi/n$, where $j = 0, 1, 2, \ldots$. Near the attractor, where $\phi \approx \pm j2\pi/n$ and $v_\phi \ll 1$, oscillatory motions of ϕ result. For a sufficiently low value of the product $\eta = kb$ and high speed $v_\phi \sim c \equiv 1$, ϕ grows linearly with time. Thus the equation of motion for $\phi(t)$ can be solved approximately for two regimés that are disconnected in the phase space.

As in the Randall-Sundrum model, there are two kinds of motion in this modified scenario. Namely, a particle either performs a unidirectional motion across the whole extra dimension or oscillatory motion in the vicinity near the points $\phi \approx \pm j2\pi/n$. The critical condition separating these two regimés can be determined under the condition in which $\dot{\phi}_0 = 0$ at $\phi = \pm j2\pi/n$ ($j = 0, 1, 2, \ldots$), where the corresponding energy is

$$E_0 = \frac{e^{-2kb}}{\sqrt{1-\dot{x}_0^2}}. \tag{5.47}$$

Figure 5.5: The phase plot for $v_\phi = be^{kbcosn\phi}\dot{\phi}$ versus ϕ with $n = 3$, $k = 10$ and $b = 0.01$.

Thus, if the energy of a particle exceeds E_0, then the particle could get across these potential barriers and thus can travel to everywhere in the extra dimension; otherwise, the particle would perform oscillatory motions.

Suppose $E < E_0$. Without loss of generality, only those periodic motions around $\phi = 0$ will be of interest in the following discussion. An approximate solution for which $n\phi \ll 1$ can be obtained with the assumption that $\phi(t) \sim \sin\left(\omega t + \alpha\right)$. Substituting the sinusoidal solution into the equation of motion and keeping only the lowest order of harmonics of the periodic motion, one finds

$$\phi(t) = \phi_0 \cos\omega_n t + \frac{\dot{\phi}_0}{\omega}\sin\omega_n t = \varphi_n \sin\left(\omega_n t + \alpha_n\right), \tag{5.48}$$

where

$$\varphi_n = \sqrt{\phi_0^2 + \frac{\dot{\phi}_0^2}{\omega_n^2}}, \quad \alpha_n = \tan^{-1}\frac{\phi_0}{\dot{\phi}_0/\omega_n}, \tag{5.49}$$

and

$$\omega_n = n\left(1 + \frac{n^2 kb\phi_0^2}{2}\right)^{-\frac{1}{2}}\sqrt{\frac{ke^{-2kb}(1-\dot{x}_0^2)}{b} - \frac{kb\dot{\phi}_0^2}{2}}.$$

Thus the period of motion in the extra dimension is roughly inversely proportional to the degree of inhomogeneity n. An approximation for $x(t)$ up to the lowest order of harmonics can be found as follows. Eq. (5.48) is put back into Eq. (5.44). From the Jacobi-Anger expansions

$$\begin{aligned} \cos(z\sin\theta) &= J_0(z) + 2\sum_{m=1}^{\infty} J_{2m}(z)\cos 2m\theta, \\ \sin(z\sin\theta) &= 2\sum_{m=0}^{\infty} J_{2m+1}(z)\sin\left[(2m+1)\theta\right], \end{aligned} \tag{5.50}$$

and

$$e^{-z\cos\theta} = I_0(z) + 2\sum_{m=1}^{\infty}(-1)^m I_m(z)\cos m\theta, \tag{5.51}$$

where $(I_m(z))$ $J_m(z)$ is the m-th (modified) Bessel functions of the first kind. Eq. (5.44) can then be expanded into harmonic components. By assuming the trial function $x \sim A + Bt + Ct^2 + D\cos(Et + F)$ and letting it satisfy the equation of motion, one finds, in the short time limit $t^{-1} \gg e^{2kb\cos n\phi_0} b\dot{b}\dot{\phi}_0^2 + k\dot{b}(1-\dot{x}_0^2)\cos n\phi_0$,

$$x(t) = x_0 - \delta x_n \cos 2\alpha_n + \left(\dot{x}_0 + 2\omega_n \delta x \sin 2\alpha_n\right)t + \lambda_n t^2 + \delta x_n \cos\left(2\omega_n t + 2\alpha_n\right) \tag{5.52}$$

where

$$\lambda_n = \frac{1}{4} b\dot{b}\dot{x}_0 \left\{ \varphi_n^2 \omega_n^2 e^{2kbJ_0(n\varphi_n)} \left[I_0\left(\zeta_n\right) - I_1\left(\zeta_n\right)\right] + \frac{2kJ_0(n\varphi_n)(1-\dot{x}_0^2)}{b} \right\}$$

with $\zeta_n = -4kbJ_2(n\varphi_n)$, and

$$\delta x_n = -\frac{1}{8} b\dot{b}\dot{x}_0 \left\{ \varphi_n^2 e^{2kbJ_0(n\varphi_n)} \left[I_0\left(\zeta_n\right) - 2I_1\left(\zeta_n\right)\right] + \frac{4kJ_2(n\varphi_n)(1-\dot{x}_0^2)}{b\omega_n^2} \right\},$$

which characterizes the fluctuation of $x(t)$. We note that in this regimé, the period of fluctuation is given by π/ω_n, which is half of that of ϕ. This is consistent with the result in the Randall-Sundrum scenario. Moreover, the amplitude δx_n of the fluctuations does not depend on the degree of inhomogeneity n.

On the other hand, if the energy of the particle $E > E_0$, then ϕ grows linearly with time. Similar techniques described above and the identity

$$e^{-z\cos\theta}\sin\theta = \frac{2}{z}\sum_{m=1}^{\infty} m(-1)^{m+1} I_m(z)\sin m\theta$$

can be used to obtain an approximation of $\phi(t)$. From numerical experiment, $\phi(t) \sim A + Bt + C\sin(Dt+E)$. If $C \geq 1$ (*i.e.* $\gamma_n \geq 1$), the equation of motion is highly non-linear and no simple analytic expressions for the constants A, B, C, D and E can be made. In the circumstances where $C \ll 1$, $\phi(t)$ up to the lowest order harmonics can be written as:

$$\phi(t) = \overline{\phi} + \overline{\dot{\phi}}t + \frac{\gamma_n}{n}\left(\sin\left(n\overline{\dot{\phi}}t + n\overline{\phi}\right) - \sin n\overline{\phi}\right), \tag{5.53}$$

where

$$\gamma_n = \frac{(1-\dot{x}_0^2)I_1(2kb) + 2kb^3\dot{\phi}_0^2}{(4kb\cos n\phi_0 - 1)b^2\dot{\phi}_0^2},$$

and

$$\overline{\phi} = \phi_0\left(1 - \frac{\gamma_n}{n}\sin n\phi_0\right) \qquad \text{and} \qquad \overline{\dot{\phi}} = \dot{\phi}_0\left(1 - \gamma_n \cos n\phi_0\right).$$

The respective $x(t)$ in the lowest order of harmonics is

$$\begin{aligned} x(t) =& x_0 - \delta x' \cos n\overline{\phi} + \left[\dot{x}_0 + n\overline{\dot{\phi}}\delta x' \sin n\overline{\phi}\right] t + \frac{b\dot{b}\overline{\dot{\phi}}^2 \dot{x}_0 (2+\gamma_n^2) I_0(-2kb)}{4} t^2 \\ &+ \delta x' \cos\left(n\overline{\dot{\phi}} t + n\overline{\phi}\right), \end{aligned} \tag{5.54}$$

where

$$\delta x' = -\frac{\dot{x}_0 b\dot{b}}{n^2}\left[2\gamma_n I_0(-2kb) - (2+\gamma_n^2) I_1(-2kb) + \frac{k}{b\overline{\dot{\phi}}^2}\left(1-\dot{x}_0^2\right)\right] \tag{5.55}$$

characterizes the fluctuations. We note that in this regimé, the period of fluctuations is equal to that of ϕ (which is also consistent with the previous results), but the amplitude of fluctuations is roughly proportional to $1/n^2$.

5.5 Summary

To conclude, particle trajectories in the Randall-Sundrum scenario are calculated analytically. As in Chapter 6, we have also assumed that a 4*D* observer can see the 5*D* particle regardless of its position in the fifth dimension. Since the fifth dimension is warped, the particle either performs a unidirectional motion across the whole extra dimension, or oscillates in the vicinity around the visible brane. In both types of motion, the trajectories projected into the 4*D* ordinary space-time contain oscillatory components, whose period is equal to the time needed to travel around the extra dimension.

In order to make connection with the physical situations, we made use of classical wave packets in the extra dimensional space-time and compared their evolution with those 4*D* space-time. It is shown that the usual wave packet in 4*D* can be reproduced, with additional t- and t^2-dependent terms, which provide a basis of the experimental determinations of the model parameters.

Finally, the effect of the inhomogeneity of the fifth dimension is also examined. We conclude that the stronger the inhomogeneity is, the shorter the period of fluctuation of the 4*D* trajectories is.

Chapter 6

Comments on Extra Forces as the Source of Quantum Fluctuations

Recall the motivation to examine extra forces in Chapters and for different kinds of extra dimensional metrics was to seek to possibility of interpreting quantum fluctuations as a consequence of extra dimensional motions. We did so by simulating the spread of the classical wave packet in extra dimensional world and comparing it with that of a quantum wave packet.

It is definitely not a new idea in seeking an interpretation for the quantum fluctuations in classical theories. Bohm [51] suggests that the position of a particle must be deterministic, but its velocity is described by a wave function satisfying the Schrodinger Equation. This “pilot wave” configuration is able to predict, for example, the result of a double-slit experiment and the collapse of wave function [52]. Bohm’s theory is actually a hidden variable theory [53], which asserts that non-locality in quantum mechanics is due to hidden freedoms which can never be observed in laboratory. Since Bell’s inequality [54] proves that local hidden variable theory cannot exist, Bohm’s theory can only be a non-local hidden variable theory if it is true.

The extra coordinate r_b or ϕ used in the previous chapters are examples of hidden variables. Their effects in the ordinary space-time can only be observed in terms of integrals over the whole range of the extra coordinate. The extra coordinate obviously can only be a local hidden variable. It implies that it would be impossible to simulate all the quantum phenomena with extra dimensions. But what we have done in this thesis is trying to provide a fully deterministic origin of microscopic fluctuations. It is by no means an attempt to replace the quantum description. The interpretation of Bell's inequality or vacuum polarization in terms extra dimensional physics would deserve investigation.

Chapter 7

Particle-Antiparticle Pairs as an Observable Effect of Extra Force

It has been a classic result in the Kaluza-Klein theory of electromagnetism that the Lorentz force law can be interpreted as the extra force when a $5D$ particle is moving in the $5D$ Kaluza-Klein space-time [3, 12]. As we have seen in Section 2.2, the electric charge of a particle can be defined from the canonical fifth momentum. This has an interesting implication because the sign of the electric charge may be related to the direction of motion in the extra dimension. That is, a pair of particle and antiparticle (two oppositely charged particles of the same mass) detected by a $4D$ observer could be the very same particle with different modes of unobserved extra dimensional motion.

In this chapter, we want to explore the possibility of introducing antiparticles based on the kinematics of a particle moving in a higher dimensional space-time. To make the discussion self-contained, we first review the arguments, due to Paul M. Dirac and Richard Feynman, that naturally leads to the existence of antiparticles, and subsequently we review the definitions of the charge, parity and time reversal operations in quantum field theory. We then review the definition of these operations in the classical theory of general relativity. Then we will derive the appropriate operations in $5D$ Kaluza-Klein theory and try to apply these operations in the combined model of Kaluza-Klein's and Randall-Sundrum's. We find that the combined model contains neutral particles that are very similar to neutrinos, which are chargeless and have tiny masses (< 1 eV).

7.1 Reasons for Antiparticles

The existence of antiparticles is one of the most successful predictions in theoretical physics [45]. In order to solve the negative energy problem in the Klein-Gordon equation, Dirac proposed the Hole Theory, in which a sea of electrons fills up all negative energy levels according to the Pauli exclusion principle. When an electron is excited from a state of negative energy to another of positive energy, the vacated negative energy level (hole), called the positron, appears as an electron with positive energy as well as positive electric charge.

Dirac proposed another relativistic equation that bears his name, and his equation is free from negative probability density by construction. It turns out that Dirac's equation is appropriate for describing particles with spin-$\frac{1}{2}$. Antiparticles are already built-in in this equation, and their existence is a natural consequence of the merging of quantum mechanics with special relativity [55].

Antiparticles play an important role in elementary processes such as particle creation and annihilation in the microscopic world, where quantum description is needed. The operation that changes a particle into its antiparticle is given by the charge conjugation which flips the sign of the charge. It can be shown that an antiparticle can be thought of as travelling in the direction of time that is reversed with respect to that of its particle [55]. This is completely legitimate in the microscopic world. There have also been some literature trying to obtain antiparticles from classical physics by making use of the negative solution of the proper time in the theory of general relativity [46].

Feynman [55] gives a heuristic argument why antiparticles are the direct consequence of a relativistic quantum theory. In his path integral formulation of quantum mechanics, a particle can freely propagate between two points with any values of magnitudes of the momentum. In particular, since quantum mechanics requires mathematically a complete set of states as basis, it is inevitable that a particle can propagate with a speed greater than the speed of light, or in other words, its world line may lie outside the light cone. Consider a particle moving in a space with a potential U. The second order perturbation can be represented by a scattering process where the particle interacts with U twice at the vertices shown in Fig. 7.1. The intermediate particle travels from (t_1,x_1) to (t_2,x_2) and the event occurs outside the light-cone (a space-like event). In the experiment, we say that the particle travels to (t_1,x_1), where it is scattered by U. Then it travels to (t_2,x_2) and is scattered again by U. In Fig. 7.2, we can transform the space-like interval in Fig. 7.1 into another space-like interval by a Lorentz transformation, after which the time direction is reversed. So far only the time is reversed. To be consistent with our intuition, we need also to reverse the parity with the particle travelling from (t_2,x_2) to (t_1,x_1), as in Fig. 7.3. This is completely admissible as long as the particle is chargeless since we have no way to distinguish the direction of the time arrow microscopically. However, if the particle has a charge q, then Fig. 7.2 and Fig. 7.3 can be made equivalent if the electric charge is conjugated. As a result, we would see in the experiment that a pair of particle and its antiparticle with the same mass but opposite electric charges are created at (t_2,x_2) (pair creation) and then the antiparticle traveling to (t_1,x_1) is annihilated by the particle (pair annihilation). Thus antiparticles are the natural consequence of the Lorentz invariance of the wave equations in the quantum theory.

In the above argument, we have made use of three inherent operations: the charge conjugation $\mathscr{C}$, the parity $\mathscr{P}$ and the time reveral $\mathscr{T}$, and they are related by the equality $\mathscr{P}\mathscr{T}=\mathscr{C}$. We can also study the effects of the operations $\mathscr{C}$, $\mathscr{P}$, $\mathscr{T}$ on the photon [56]. Our goal is to see their effects on the scalar and vector potentials. First of all, from the Gauss Law

$$\nabla\cdot\mathbf{E}(t,\mathbf{x})=\rho(t,\mathbf{x}).$$

Under the parity operation $\mathscr{P}$, $\rho(t,\mathbf{x})\to\rho(t,-\mathbf{x})$ and $\nabla\to-\nabla$. Thus, in order to

Figure 7.1: Second order scattering of a particle by an external potential U. The intermediate particle goes from (t_1, x_1) to (t_2, x_2) with $t_2 > t_1$.

Figure 7.2: Second order scattering of a particle by an external potential U. The event is Lorentz- transformed from Fig. 7.1. The intermediate particle goes from (t_1, x_1) to (t_2, x_2) with $t_1 > t_2$.

Figure 7.3: An equivalent description of the event in 7.2 by a charge conjugation when the particle has a charge q. The intermediate particle now goes from (t_2, x_2) to (t_1, x_1) with $t_1 > t_2$.

preserve the Gauss Law, we require

$$\mathscr{P} : \mathbf{E}(t, \mathbf{x}) \to -\mathbf{E}(t, -\mathbf{x}).$$

In terms of the scalar and vector potentials, **E** can be expressed as

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t},$$

which implies that

$$\mathscr{P} : \Phi(t, \mathbf{x}) \to \Phi(t, -\mathbf{x}), \qquad \mathbf{A}(t, \mathbf{x}) \to -\mathbf{A}(t, -\mathbf{x}). \tag{7.1}$$

Similar arguments show that

$$\mathscr{C} : \Phi(t, \mathbf{x}) \to -\Phi(t, \mathbf{x}), \qquad \mathbf{A}(t, \mathbf{x}) \to -\mathbf{A}(t, \mathbf{x}); \tag{7.2}$$

$$\mathscr{T} : \Phi(t, \mathbf{x}) \to \Phi(-t, \mathbf{x}), \qquad \mathbf{A}(t, \mathbf{x}) \to -\mathbf{A}(-t, \mathbf{x}). \tag{7.3}$$

Lastly, we note that

$$\mathscr{P}\mathscr{T} A^\mu(t, \mathbf{x}) = A^\mu(-t, -\mathbf{x}) \quad \text{and} \quad \mathscr{C} A^\mu(t, \mathbf{x}) = -A^\mu(t, \mathbf{x}). \tag{7.4}$$

where $A^\mu = (\phi, \mathbf{A})$. Since $\mathscr{P}\mathscr{T} = \mathscr{C}$, we must have $A^\mu(-t, -\mathbf{x}) = -A^\mu(t, \mathbf{x})$. Furthermore,

$$F^\mu{}_\nu(-t, -\mathbf{x}) = F^\mu{}_\nu(t, \mathbf{x}). \tag{7.5}$$

7.2 Attempts in Classical Physics

Costella *et al.* [46] show that antiparticles may also come out naturally in classical mechanics. First of all, they notice that in the four dimensional theory of special relativity, the proper time is defined through the line element $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, where $\eta_{\mu\nu}$ is the Minkowski metric diag$\{1,-1,-1,-1\}$. Obviously, the proper time can only be defined up to a sign:

$$ds = \pm\sqrt{\eta_{\mu\nu} dx^\mu dx^\nu}. \tag{7.6}$$

Conventionally, we take the positive root such that the proper time runs in the same direction as the coordinate time t and that the particle always gets older. For example, for a stationary particle, $ds = dx^0 \equiv dt$ as required by our daily experience. On the other hand, Costella *et al.* conjecture that a particle whose proper time ds runs in the opposite sense of dt is seen as an antiparticle with respect to "us". They note that there is no mechanism to change a particle into an antiparticle since they are separated in disjoint regions in the forward and backward lightcones. To see how their conjecture can be implemented in the classical theory, they introduced a set of operations. Intuitively, the time reversal and parity operations can be defined as

$$\begin{aligned} &\mathscr{T} : t \to -t, \\ &\mathscr{P} : x^i \to -x^i. \quad (i = 1,2,3.) \end{aligned} \tag{7.7}$$

With their conjecture, they further define the charge conjugation as

$$\mathscr{C} : s \to -s. \tag{7.8}$$

If the charge conjugation $\mathscr{C}$ is applied to the geodesic equation, then one gets

$$\frac{d^2x^\mu}{d(-s)^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d(-s)} \frac{dx^\beta}{d(-s)} = 0, \tag{7.9}$$

which is just the original geodesic equation. Thus the antiparticle follows the same trajectory as the particle. On the other hand, the Lorentz force law in a flat space reads

$$\frac{d^2x^\mu}{ds^2} = qF^\mu{}_\nu \frac{dx^\nu}{ds}. \tag{7.10}$$

If $\mathscr{C}$ is applied to both sides of the Lorentz force law, one gets

$$\frac{d^2x^\mu}{d(-s)^2} = qF^\mu{}_\nu \frac{dx^\nu}{d(-s)} \tag{7.11}$$

or, equivalently,

$$\frac{d^2x^\mu}{ds^2} = -qF^\mu{}_\nu \frac{dx^\nu}{ds}. \tag{7.12}$$

Therefore, this equation of motion corresponds to an equivalent particle with negated charge, which is just the antiparticle.

We also want to apply the combined operation $\mathscr{P}\mathscr{T}$ to the Lorentz force law. Clearly, $\mathscr{P}\mathscr{T}$ is the reversal of the four vector x^μ:

$$\mathscr{P}\mathscr{T} : x^\mu \to -x^\mu. \tag{7.7'}$$

In order that the Maxwell equations be invariant under the operation (7.7'), we must have [57]

$$\mathscr{P}\mathscr{T} : F^\mu{}_\nu(x^\gamma) \to -F^\mu{}_\nu(-x^\gamma). \tag{7.13}$$

Applying this operation on the both sides of Eq. (7.10), we get

$$-\frac{d^2x^\mu}{ds^2} = qF^\mu{}_\nu(-x^\gamma)\frac{dx^\nu}{ds}. \tag{7.14}$$

We note that $\mathscr{P}\mathscr{T} = \mathscr{C}$ only if A^μ is an even function, as in Eq. (7.5).

7.3 5*D* Kaluza-Klein Scenario

As pointed out, the Lorentz force can be regarded as an extra force in the 5*D* Kaluza-Klein space-time. Then we should be able to define similar operations (charge, parity and time reversal) in a higher dimensional model. Our motivation comes from the fact that in a flat compactified extra dimension, once the direction of positive ϕ is defined, the particle may go either along or opposite to the direction of ϕ. We are interested in the following question: if the compactified extra dimension is not homogeneous such that the direction of ϕ becomes important, may we distinguish particles and antiparticles from the directions of their extra dimensional motion? This question at least makes some sense in 5D Kaluza-Klein electromagnetism, where the charge of a free particle can be defined from the particle's kinematics.

The 5D Kaluza-Klein theory has been reviewed in Chapter 2. It starts with the well-known metric which can be rewritten in the following form:

$$d\hat{s}^2 = g_{\mu\nu}dx^\mu dx^\nu - \left(A_\mu dx^\mu + r_b d\phi\right)^2, \tag{7.15}$$

where κ is absorbed into the vector potential A_μ and the size of the extra dimension r_b is written out explicitly. The $U(1)$ gauge symmetry of the metric (7.15) can be shown as follows. If the fifth coordinate is shifted by a function $\Lambda(x^\gamma)$:

$$\phi \to \phi + r_b^{-1}\Lambda(x^\gamma), \tag{7.16}$$

then the metric becomes:

$$d\hat{s}^2 \to g_{\mu\nu}dx^\mu dx^\nu - \left\{\left[A_\mu + \partial_\mu\Lambda(x^\gamma)\right]dx^\mu + r_b d\phi\right\}^2. \tag{7.17}$$

Thus the metric is invariant if A_μ transforms as

$$A_\mu \to A_\mu - \partial_\mu\Lambda(x^\gamma). \tag{7.18}$$

This $U(1)$ gauge symmetry is due to the isometry of the extra dimension [4].

The equations of motion under the metric (7.15) are [12]

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = a_\phi F^\mu{}_\nu\frac{dx^\nu}{ds}, \tag{7.19}$$

$$A_\nu\frac{dx^\nu}{ds} + r_b\frac{d\phi}{ds} = -a_\phi, \tag{7.20}$$

where a_ϕ is the canonical velocity along the extra dimension and is a constant of motion, and $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}. \tag{7.21}$$

Eq. (7.19) is analogous to the Lorentz force law. Thus, in the 5D Kaluza-Klein metric, the extra force appears as the 4D Lorentz force law. For simplicity, $g_{\mu\nu} = \eta_{\mu\nu}$ will be taken.

In STM theory, an extra force f^μ ($\mu = t,x,y,z$) is named as an "anomalous fifth force" if its scalar product with the four velocity u^μ does not vanish: $f^\mu u_\mu \neq 0$, implying that f^μ must be a non-gravitational force. The Lorentz force in Eq. (7.19) satisfies $f^\mu u_\mu = 0$. Thus it is not an anomalous fifth force.

The equations of motion also imply that when $\mu = 0$,

$$\frac{d}{ds}\left(\frac{dt}{ds} + a_\phi A_\mu\right) = a_\phi A_{\nu;0}\frac{dx^\nu}{ds} \qquad \text{and} \qquad \frac{dt}{ds} = \frac{1}{\sqrt{1-v^2}},$$

where $v = \sqrt{\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}$ is the magnitude of the 3-velocity, δ_{ij} being the Kronecker delta. As a consequence, we may define the energy of a Kaluza-Klein particle as

$$E_{\text{KK}} = \frac{M_5}{\sqrt{1-v^2}} + M_5 a_\phi A_0, \tag{7.22}$$

where M_5 is the $5D$ rest mass.

By comparing either Eq. (7.19) to the Lorentz force law or Eq. (7.22) to the Hamiltonian of a particle in an electromagnetic field, one may want to define the charge-to-mass ratio as

$$\frac{q}{m_0} = a_\phi, \tag{7.23}$$

where m_0 is the $4D$ rest mass. In Chapter 3, we found the relation between the $4D$ rest mass m_0 and the $5D$ rest mass M_5 under the canonical metric. Similar calculations give the relation between the $4D$ and $5D$ rest masses under the Kaluza-Klein metric (7.15) as follows:

$$m_0 = \frac{M_5}{\sqrt{1 - r_b^2\left(\frac{d\phi}{ds}\right)^2}}. \tag{7.24}$$

We now consider the weak field limit:

$$A_\nu \frac{dx^\nu}{ds} \ll r_b \frac{d\phi}{ds}. \tag{7.25}$$

In this limit, $a_\phi \approx -r_b(d\phi/ds)$. In other words, the velocity in the extra dimension ($v_\phi = r_b d\phi/ds$) and m_0 are approximately constants. The Lorentz force law (7.19) and the charge-to-mass ratio in Eq. (7.23) can now be rewritten respectively as

$$\frac{d^2x^\mu}{ds^2} = -r_b F^\mu{}_\nu \frac{d\phi}{ds}\frac{dx^\nu}{ds}, \tag{7.19'}$$

and

$$\frac{q}{m_0} = -r_b \frac{d\phi}{ds}. \tag{7.23'}$$

That is, in the weak field limit (7.25), the charge-to-mass ratio is completely determined by the kinematics in the extra dimension.

We are ready to define appropriate $\mathscr{C}$, $\mathscr{P}$, and $\mathscr{T}$ operations in the $5D$ Kaluza-Klein theory. The $\mathscr{P}$ and $\mathscr{T}$ operations are the same as in Eq. (7.7):

$$\begin{aligned} &\mathscr{T} : t \to -t, \\ &\mathscr{P} : x^i \to -x^i. \quad (i = 1,2,3.) \end{aligned} \tag{7.7}$$

The charge conjugation can be defined as

$$\mathscr{C} : \phi \to -\phi. \tag{7.26}$$

Applying this operation to Eq. (7.23'), one gets

$$\mathscr{C}\left(\frac{q}{m}\right) = \mathscr{C}\left(-r_b \frac{d\phi}{ds}\right) = r_b \frac{d\phi}{ds} = -\frac{q}{m}. \tag{7.27}$$

The application of the composite operation $\mathscr{P}\mathscr{T}$ to Eq. (7.19') results in the same equation (7.14). Thus we get an invariant Lorentz force law if Eq. (7.5) holds.

7.4 Generalizations to Brane Theories

Kaluza-Klein theory is not a brane theory; indeed, the $U(1)$ gauge field itself propagates through the fifth dimension. In brane theories, in contrast, the photon is introduced from an external Lagrangian, which has nothing to do with the underlying geometry. It seems that these two theories are mutually exclusive, and we ask if we could unify them into one single metric. There is a fundamental difficulty, namely that the $U(1)$ gauge symmetry inherent in the Kaluza-Klein theory is lost in the Randall-Sundrum metric because of the warp factor. We thus make assumptions below so that the features of both Randall-Sundrum and Kaluza-Klein models can be retained: 1. Two branes are located at $\phi = 0$ and $\phi = \pi$, between which is an anti-de Sitter space-time; 2. When the electromagnetic field is weak enough, the five-dimensional space-time is well described by Randall-Sundrum metric; and, 3. The electromagnetic field is coupled with the gravitational field via Kaluza-Klein mechanism.

Therefore, the metric we propose to merge the Kaluza-Klein and Randall-Sundrum ones is as follows:

$$d\hat{s}^2 = e^{-2kr_b|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu - \left(e^{-2kr_b|\phi|}A_\mu dx^\mu + r_b d\phi\right)^2. \qquad (7.28)$$

The size of the extra dimension r_b is assumed to be constant. The $U(1)$ gauge symmetry described by the transformations (7.16) and (7.18) will no longer be applicable. This broken gauge symmetry also exists in STM theory when the canonical gauge is used [12]. The equations of motion under this metric are

$$\frac{d}{d\hat{s}}\left\{e^{-2kr_b|\phi|}\eta_{\mu\nu}\frac{dx^\nu}{d\hat{s}} + e^{-2kr_b|\phi|}A_\mu \hat{p}_5\right\} = e^{-2kr_b|\phi|}\hat{p}_5 A_{\gamma,\mu}\frac{dx^\gamma}{d\hat{s}} \qquad (7.29)$$

and

$$\frac{d\hat{p}_5}{d\hat{s}} = ks_\phi e^{-2kr_b|\phi|}\eta_{\mu\nu}\frac{dx^\mu}{d\hat{s}}\frac{dx^\nu}{d\hat{s}} - ks_\phi e^{-2kr_b|\phi|}\hat{p}_5\, A_\nu \frac{dx^\nu}{d\hat{s}}, \qquad (7.30)$$

where

$$\hat{p}_5 = e^{-2kr_b|\phi|}A_\nu \frac{dx^\nu}{d\hat{s}} + r_b\frac{d\phi}{d\hat{s}}. \qquad (7.31)$$

We will assume again that the field A_μ is weak enough that the second or higher powers of A_μ can be neglected.

By finding the equation of motion for (d^2x^μ/ds^2), where s is the $4D$ proper time, we obtain a kinetic term from Eq. (7.29) that is responsible for the electromagnetic interaction:

$$r_b F^\mu{}_\gamma \frac{dx^\gamma}{ds}\frac{d\phi}{ds}, \qquad (7.32)$$

which has the same form as Eq. (7.19). In the weak field limit, this term can be calculated by using the trajectory of $\phi(t)$ in the absence of A_μ. Such trajectory has been found in Chapter 6, and it has essentially two types of motions: (1) the unidirectional motions across the whole extra dimension and (2) the oscillatory motions in the vicinity of the visible brane, depending on the energy of the particle. Eq. (7.32) implies that one may define a charge-to-mass ratio for the particle:

$$\frac{q}{m_0} = r_b \frac{d\phi_0}{ds}, \tag{7.33}$$

where $\phi_0(t)$ is given by Eq. (5.15), which is the function $\phi(t)$ in the absence of A_μ, and m_0 is the 4D rest mass related to the 5D rest mass M_5 by Eq. 5.28

$$m_0 = \frac{e^{-kr_b|\phi|}M_5}{\sqrt{1 - r_b^2 e^{2kr_b|\phi|}(\frac{d\phi}{ds})^2}} \,. \tag{5.28}$$

Notice that the expression of m_0 is very similar to the definition of energy E given by Eq. (5.6) in the RS braneworld scenario. Indeed, in the limit $\dot{x} \ll 1$, whence $ds \approx dt$ and, m_0 is equivalent to E. Most importantly, m_0 is a constant throughout the motion and is determined once the initial conditions are specified. The charge of the particle can now be defined as

$$q = M_5 r_b e^{-kr_b|\phi_0|} \frac{d\phi_0}{ds} \left[1 - r_b^2 e^{2kr_b|\phi_0|} \left(\frac{d\phi_0}{ds}\right)^2\right]^{-\frac{1}{2}} . \tag{7.34}$$

We remark that our calculation is slightly different from Klein's (but similar to the calculation in STM theory [12]). Apart from the factor $\kappa \equiv \sqrt{16\pi G}$, Klein relates the size of the extra dimension and the Planck constant to the electric charge via de Broglie's relation (cf. Eq. (2.21)), whereas we relate the 5D mass and the warp factor to the electric charge via the Lorentz force law (cf. Eq. (7.32)). In the former case, quantum effect of the compact extra dimension is involved. In contrast, the present work aims to investigate the possible explanation of charge in terms of particle kinematics only.

There are three possibilities for the charge of a particle as defined by Eq. (7.34):

1. Unidirectional motion with $\frac{d\phi_0}{ds} > 0$;
2. Unidirectional motion with $\frac{d\phi_0}{ds} < 0$;
3. Oscillatory motion such that the charge-to-mass ratio is rapidly changing its sign.

Similar to the discussion in the last section, the first two possibilities may be the definitions of particles and antiparticles. At the beginning of each experiment, either Possibility 1 or 2 is taken to be the definition of "positive charge" and the other be "negative charge", and vice versa. Since these particles perform unidirectional motion in the extra dimension, their total energies must be greater than the critical energy (5.7). In other words, their 4D rest masses m_0 must be at least of order of the 5D rest mass: $m_0 \gtrsim M_5$. Possibility 3 may be responsible for a neutral particle since the sign of the charge is rapidly flipping (period $\tau_x \sim e^{-kr_b\pi}/ck$) with the average zero. Their 4D rest mass must be less than the 5D rest mass: $m_0 < M_5$. We assume that the same kind of particles with the same M_5 are engaged in all of the three possibilities.

Consider two critical cases: (1) the particle has just enough energy to perform unidirectional motions. In other words, $E = E_{\rm cr}$ such that $d\phi_0/ds = 0$ at $\phi_0 = 0$; (2) the particle oscillates around the visible brane ($\phi_0(t) \approx \pi$).[1] In the former case, $m_0 = M_5$ whereas in the latter case, $m_0 \approx e^{-kr_b\pi}M_5$. We thus arrive at the following conclusion: *there is a mass hierarchy between charged and neutral particles in the combined Kaluza-Klein-Randall-Sundrum model*.

Such mass hierarchy does exist in the Standard model of the leptons, in which the neutrinos are chargeless and are believed to have extremely tiny masses. To date, only their upper bounds can be obtained, which are highly sensitive to the experimental methods [2] . In the laboratory, tritium β-decay experiments put a limit on the mass of electron-neutrino: $m_{\nu_e} < 3$ eV; measurements of muon momentum in pion-decay experiments put a limit on the mass of muon-neutrino $m_{\nu_\mu} < 0.19$ MeV; and tau-decay experiments put a limit on the mass of tau-neutrino $m_{\nu_\tau} < 18$ MeV [58]. More stringent limits are found from cosmological observations (Refs. [59, 60] and references therein). These observations are sensitive to the sum of all light neutrino masses, thus providing the upper bounds for each of them. Recently,

[1] In both cases, we keep $\dot{x} \ll 1$.

[2] The weak eigenstates and the mass eigenstates are not distinguished in this context.

Goobar *et al.* [59] derived from the most recent data of the cosmic microwave background (CMB), the large scale structure of galaxies, and Type Ia supernovae that the sum of all light neutrino masses is $\Sigma m_\nu \leq 0.62$ eV (95% C.L.), and they also derived from the Lyman-α forest that the sum is $\Sigma m_\nu \leq 0.2 - 0.4$ eV (95% C.L.). Fukugita *et al.* [60] also constrained from the WMAP-3 data the neutrino mass $\Sigma m_\nu < 2$ eV by assuming the flat ΛCDM model with power-law adiabatic perturbations. In any case, cosmological observations suggest that the neutrino masses are of the order less than 1 eV. But this poses a mass scale hierarchy between the neutrinos and the charged leptons, whose masses are of order $1 - 1000$ MeV.

In $4D$ theories, the see-saw mechanism has been used to solve the neutrino mass problem, in which the neutrinos are chiral, and the neutrino mass is suppressed by the ratio of the Dirac mass to the Majorana mass terms. The right-handed neutrinos are naturally decoupled from other matters, thus explaining the parity asymmetry [61]. On the other hand, in the Randall-Sundrum scenario, the neutrino mass is suppressed by the coupling between the non-chiral, left-handed neutrinos localized on the brane and a right-handed bulk fermion field. The suppression is provided by the large overlap (due to the warp factor) of the amplitudes of the bulk fermion and the brane leptons at the visible brane [11]. In the Kaluza-Klein-Randall-Sundrum model of this work, on the other hand, a possible explanation of the neutrino mass problem is given by the different types of the extra dimensional motion of a single particle, and the suppression of the $4D$ rest mass is given by the warp factor. This result is remarkable because this model has raised the possibility that the charged leptons and their neutrino partners (plus their antiparticles) are actually represented by the same particle in a $5D$ world.

To connect the above idea to nature, we set M_5 in Eq. (7.34) to be the electron mass, which is 0.511 MeV [58], and we set q to be the electron charge e. Notice that $\phi_0(t)$ and $d\phi_0/ds$ are periodic functions, and so are the charge-to-mass ratios defined in Eq. (7.34). However, the periods of motion in the extra dimension are too short that only the time average of the charge is measurable. We take the non-relativistic limit $\dot{x} \ll 1$, whence we have $ds \approx dt$. Then the time-average can be evaluated by using the solution (5.15). We choose the initial conditions to be $\phi_0 = 0$ and $v_{\phi_0} = 0^+$. Then the period of extra dimensional motion is $\tau_\phi = 2e^{kr_b\pi}/k$. Since $m_0 = M_5$ in this situation, the time-average of the electric charge is (in Gaussian units)

$$\langle q \rangle_t = \frac{\ln(1+4\,e^{2kr_b\pi})}{4\,e^{kr_b\pi}}\kappa M_5. \tag{7.35}$$

where we have put back $\kappa = \sqrt{16\pi G}$, G being the 4D gravitational constant, for dimension consistency. The 4D charge-to-mass ratio is then given by

$$\frac{\langle q \rangle_t}{m_0} = \frac{\ln(1+4\,e^{2kr_b\pi})}{4\,e^{kr_b\pi}}. \tag{7.36}$$

which is independent of M_5 and is a function of kr_b only. Suppose we set m_0 to be the electron mass $m_e = 0.511$ MeV and $\langle q \rangle_t$ the electron charge $e = 4.8 \times 10^{-10}$ esu, then from Eq. (7.36), we can obtain the value of kr_b by putting back the numerical values of q_e and κ. This gives

$$kr_b = 2.66\,. \tag{7.37}$$

Thus, to get back the empirical charge-to-mass ratio of an electron requires a significantly smaller warp factor than the one needed to explain the Hierarchy Problem. The “Kaluza-Klein-Randall-Sundrum” (KK-RS) electron-neutrino mass in this model is given by

$$m_{\nu_e} = e^{-kr_b\pi}M_5 \tag{7.38}$$

Since $m_e = M_5$, the ratio of electron-neutrino mass to electron mass is given by the warp factor:

$$\frac{m_{\nu_e}}{m_e} = e^{-kr_b\pi} \approx 2 \times 10^{-4}, \tag{7.39}$$

which suggests a mass hierarchy between the charged leptons and the neutrinos. The empirical value is 2×10^{-6}, which is a factor 100 smaller than that in Eq. (7.39).

In Eq. (7.36), the Randall-Sundrum parameter kr_b has been used to tune the time-average of the electric charge of a bulk particle. The same parameter is also responsible for the neutrino mass in Eq. (7.39). Thus we have been trying to interpret two phenomena with only one parameter. It is desirable to introduce another parameter in the KK-RS metric so that kr_b is only responsible for the neutrino mass hierarchy, and this new parameter would fine-tune the time-average of electric charge.

Consider the following metric:

$$d\hat{s}^2 = e^{-2kr_b|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu - \left(e^{-\varepsilon kr_b|\phi|}A_\mu dx^\mu + r_b d\phi\right)^2. \qquad (7.28")$$

This metric is identical to that in Eq. (7.28) if $\varepsilon = 2$. The new parameter ε serves as a coupling of the electromagnetic field with the Randall-Sundrum field. The greater the value it is, the stronger the coupling is. In the weak field limit where second or higher order terms of A_μ can be neglected, the 4D rest mass m_0 is given by Eq. (5.28) and is independent of ε. By exactly the same argument in the above, we find that the charge-to-mass ratio can be defined as

$$\frac{q}{m_0} = r_b e^{(2-\varepsilon)kr_b|\phi_0|}\frac{d\phi_0}{ds}, \qquad (7.23")$$

where $\phi_0(t)$ is the trajectory in the extra dimension when A_μ vanishes. Then the time-average of the electric charge for a bulk particle, with the critical energy E_{cr} and $\dot{x} \ll 1$ such that it is just able to perform unidirectional motion in the extra dimension, is given by (in Gaussian units)

$$\langle q\rangle_t = \left\{\frac{1-\left(1+4e^{2kr_b\pi}\right)^{1-\frac{\varepsilon}{2}}}{2\left(\varepsilon-2\right)e^{kr_b\pi}}\right\}\kappa M_5. \qquad (7.34")$$

Eq. (7.34") reduces to Eq. (7.34) when $\varepsilon = 2$. Similar to the discussion above, we consider two critical cases where a particle is either oscillating around the visible brane, or it has just enough energy to get across the invisible brane. In the latter case, we get an electron with 4D mass $m_e = M_5$ while in the former case, we get a neutrino with 4D mass $m_{\nu_e} = e^{-kr_b\pi}M_5$.

Now we require that the mass ratio $m_{\nu_e}/m_e = e^{-kr_b\pi}$ to be the empirical ratio. Clearly, $m_e = 0.511$ MeV. We will take the cosmological upper bounds for the neutrino masses, which are of order 1 eV. Some extra dimensional models suggest that the fundamental constants might be varying with the volume of the extra dimensions as the Universe is evolving. Such variations have been constrained to $+0.1\%$ in order to explain the discrepancies in the current observations between big bang nucleosynthesis and WMAP [6]. Thus we assume that such variations have only a tiny effect on the neutrino masses, and hence we take the cosmologically measured values as the current values. Since empirically $m_{\nu_e}/m_e = 2\times 10^{-6}$, one needs $kr_b = 4.18$. Thus, as in the previous metric, in order to explain the mass hierarchy between the electron and electron-neutrino, we require a smaller warp factor than that for the Hierarchy Problem. Moreover, by requiring

$$\frac{\langle q\rangle_t}{m_e} = \frac{1-\left(1+4e^{2kr_b\pi}\right)^{1-\frac{\varepsilon}{2}}}{2\left(\varepsilon-2\right)e^{kr_b\pi}/\kappa} \tag{7.40}$$

to be the electron's charge-to-mass ratio, we find $\varepsilon = 1.55$. The required value of ε is smaller than 2, implying a weaker coupling between the electromagnetic and Randall-Sundrum fields than the previous model.

To conclude this section, we found that charged and neutral particles may correspond to two modes of extra dimensional motion moving in a $5D$ Kaluza-Klein-Randall-Sundrum world. A mass hierarchy naturally arises between the neutral and charged particles, where the ratio of the mass of the charged particle to that of the neutral particle is given by the warp factor $e^{-kr_b\pi}$. The charge-to-mass of the electron is given by its time-averaged extra dimensional motion, which depends on the curvature parameter kr_b and the coupling strength ε between the electromagnetic field and the Randall-Sundrum gravitational field. We have considered structureless particles only. If the particle has a spin $\mathbf{S}$, then the particle would show a magnetic moment [45]

$$\mu = g\frac{q}{2m_0}\mathbf{S}, \tag{7.41}$$

where g is the gyromagnetic ratio. Thus, the charge conjugation flips the sign of the magnetic moment. This conclusion applies to quarks and leptons, which have no internal structures. However, the present work does not account for the magnetic moments for neutral hadrons, such as the neutron, and mesons, which are bound states of quarks.

7.5 Summary

In this chapter, we have explored the possibility of introducing antiparticles based on the kinematics of a particle moving in a higher dimensional space-time. We have made use of the extra force, which is the electromagnetic Lorentz force, in the Kaluza-Klein model to define the electric charge of a particle moving in the 5D space-time. A set of charge-conjugation, time reversal and parity operations in the extra dimensional world have been defined. We conjecture that charged particles are the correspondence of 5D particles that are moving unidirectionally in a Kaluza-Klein-Randall-Sundrum 5D space-time, while neutral particles are those that are oscillating around the visible brane. A mass hierarchy naturally arises between the neutral and charged particles, where the ratio of the mass of the charged particle to that of the neutral particle is given by the warp factor. Thus, we have tried to explain the neutrino mass problem by tuning the curvature parameter kr_b and the coupling parameter ε between the electromagnetic field and the gravitational field. This have advantages over the previous attempts in explaining the neutrino mass in that we don't need to define a bulk fermion field which interact with the neutrinos, and we also don't need a see-saw mechanism. Rather, we have made use of the "coupling" (described by ε) between the electromagnetic field and the gravitational field, so that we can explain the electric charge and the mass hierarchy at the same time.

In the present framework, due to the $S_1/\mathbb{Z}_2$ symmetry in both Kaluza-Klein and Randall-Sundrum models, the particle should, in principle, be equally probable to be positively charged or negatively charged. However, it has long been observed in the laboratory that nature is mainly made out of particles in the low energy configurations, which is termed as the particle-antiparticle asymmetry (see, for example, [62, 63, 64] and references therein). Thus our model does not solve this problem either.

Furthermore, the present framework is not able to give an account for the existence of the three flavors of leptons. We have only applied the model to the electron flavor. Since the charges of all charged leptons are fixed at the electron charge while their masses vary, Eq. (7.34") asserts that we need different values of ε for different lepton flavor. But this is unnatural because the background gravitational field should be the same for all matters.

Chapter 8

A Summary of this Thesis

In this thesis, we have studied the classical particle trajectories in higher dimensional models. To be able to examine these higher dimensional trajectories, we have assumed that we can observe a 5D particle regardless of its position in the extra dimension.

In Chapter 6, the effect of the extra force on 4D particle trajectories under the generalized Robertson-Walker metric has been studied. We found that the extra force is constant in time (to the first order of $\dot{b}$, the rate of the change of the size of the extra dimension), and the deviations of the 4D trajectories are in general linear with time. The periodicity of the compactified extra dimensions does not show up in the deviation. The resultant spectrum of the 4D projected trajectories obtained by randomizing the initial conditions of the extra coordinates should be the one which is really observed in the laboratory. We proposed a classical test by constructing a classical wave packet in 5D Friedmann universe, and we found the additional time-dependent terms over the classical wave packet in the usual 4D space-time, which provide a basis of the experimental determinations of the model parameters.

In Chapter 6, particle trajectories in the Randall-Sundrum scenario are calculated analytically. The fifth dimension is warped, and we found two different modes of fifth dimensional motions: the particle either performs a unidirectional motion across the whole extra dimension, or oscillate in the vicinity around the visible brane. In both types of motion, periodic fluctuations of the 4*D* projected trajectories are found, whose period is equal to the time needed to travel around the extra dimension. We also compared a classical wave packet in 5*D* with that in 4*D*, and found the time-dependent correction terms. The effect of the inhomogeneity of the fifth dimension is also examined. We conclude that the stronger the inhomogeneity is, the shorter the period of fluctuation of the 4*D* trajectories is.

Finally, in Chapter 7, we have extensively examined the extra force in the Kaluza-Klein model, that is the Lorentz force law. We have made use of the fifth dimensional momentum to define the electric charge of a particle moving in the 5*D* space-time. We have defined a set of charge-conjugation, time reversal and parity operations in the higher dimensional world. We conjecture that charged particles are the correspondence of 5*D* particles that are moving unidirectionally in a Kaluza-Klein-Randall-Sundrum 5*D* space-time, while neutral particles are those that are oscillating around the visible brane. A mass hierarchy naturally arises between the neutral and charged particles, where the ratio of the mass of the charged particle to that of the neutral particle is given by the warp factor. Thus, we have tried to explain the neutrino mass problem by tuning the curvature parameter kr_b and the coupling parameter ε between the electromagnetic field and the gravitational field. This have advantages over the previous attempts in explaining the neutrino mass in that we don't need to define a bulk fermion field which interact with the neutrinos, and we also don't need a see-saw mechanism. Rather, we have made use of the "coupling" (described by ε) between the electromagnetic field and the gravitational field, so that we can explain the electric charge and the mass hierarchy at the same time.

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