Box 7-2. *(continued)*

The colatitude θ_E is stored instead of the latitude λ_E because it is θ_E that is actually used (Box 7-1).
The second subroutine (ROTATE) performs the rotation:

Function	Stack x	Stack y	Stack z	Stack t	Registers R_{11}	Registers R_{12}	Registers R_{13}
LBL "ROTATE"	r	ϕ	λ	-	θ_E	ϕ_E	Ω
RCL 12	ϕ_E	r	ϕ	λ	θ_E	ϕ_E	Ω
CHS	$-\phi_E$	r	ϕ	λ	θ_E	ϕ_E	Ω
ST + Z	$-\phi_E$	r	ϕ_1	λ	θ_E	ϕ_E	Ω
CLX	0	r	ϕ_1	λ	θ_E	ϕ_E	Ω
RCL 11	θ_E	r	ϕ_1	λ	θ_E	ϕ_E	Ω
CHS	$-\theta_E$	r	ϕ_1	λ	θ_E	ϕ_E	Ω
XEQ "ROT 2"	r	ϕ_2	λ_1	$-\theta_E$	θ_E	ϕ_E	Ω
RCL 13	Ω	r	ϕ_2	λ_1	θ_E	ϕ_E	Ω
ST+ Z	Ω	r	ϕ_3	λ_1	θ_E	ϕ_E	Ω
CLX	0	r	ϕ_3	λ_1	θ_E	ϕ_E	Ω
RCL 11	θ_E	r	ϕ_3	λ_1	θ_E	ϕ_E	Ω
XEQ "ROT 2"	r	ϕ_4	λ'	θ_E	θ_E	ϕ_E	Ω
RCL 12	ϕ_E	r	ϕ_4	λ'	θ_E	ϕ_E	Ω
ST+ Z	ϕ_E	r	ϕ'	λ'	θ_E	ϕ_E	Ω
CLX	0	r	ϕ'	λ'	θ_E	ϕ_E	Ω
RCL 13	Ω	r	ϕ'	λ'	θ_E	ϕ_E	Ω
R↓	r	ϕ'	λ'	Ω	θ_E	ϕ_E	Ω
XEQ "S-C"	x'	y'	z'	Ω	θ_E	ϕ_E	Ω
XEQ "C-S"	r	ϕ'	λ'	Ω	θ_E	ϕ_E	Ω
RETURN	r	ϕ'	λ'	Ω	θ_E	ϕ_E	Ω

The "S-C" and "C-S" conversions between spherical and Cartesian coordinates might seem to negate each other, but they perform the important function of converting ϕ' to the range $-180° \leq \phi' \leq 180°$. The operation ST+ Z, CLX was used instead of the equivalent operation "ROT 3" because it takes fewer bytes of memory and operates faster.

Box 7-3. How to Rotate Using a Computer.

If you would rather work with algebra than geometry, you can do finite rotations using a 3×3 matrix. If point **A** is a vector with global Cartesian coordinates (A_x, A_y, A_z) prior to rotation, then the components (A_x', A_y', A_z') of **A** after rotation to **A′** may be found from the matrix multiplication

$$\mathbf{A}' = \mathbf{R}\,\mathbf{A}$$

where **R** represents a 3×3 matrix. Writing all of the terms of the vector and matrix we have

(continued)

Box 7-3. *(continued)*

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Applying the usual rules for matrix multiplication gives the equations

$$A_x' = R_{11}A_x + R_{12}A_y + R_{13}A_z$$
$$A_y' = R_{21}A_x + R_{22}A_y + R_{23}A_z$$
$$A_z' = R_{31}A_x + R_{32}A_y + R_{33}A_z$$

To define the elements of the rotation matrix **R** we need to know the Cartesian coordinates of the Euler pole $\mathbf{E} = (E_x, E_y, E_z)$ and the angle of rotation Ω. The elements of the matrix are then given by

(first row of matrix)

$$R_{11} = E_x E_x (1 - \cos \Omega) + \cos \Omega$$
$$R_{12} = E_x E_y (1 - \cos \Omega) - E_z \sin \Omega$$
$$R_{13} = E_x E_z (1 - \cos \Omega) + E_y \sin \Omega$$

(second row of matrix)

$$R_{21} = E_y E_x (1 - \cos \Omega) + E_z \sin \Omega$$
$$R_{22} = E_y E_y (1 - \cos \Omega) + \cos \Omega$$
$$R_{23} = E_y E_z (1 - \cos \Omega) - E_x \sin \Omega$$

(third row of matrix)

$$R_{31} = E_z E_x (1 - \cos \Omega) - E_y \sin \Omega$$
$$R_{32} = E_z E_y (1 - \cos \Omega) + E_x \sin \Omega$$
$$R_{33} = E_z E_z (1 - \cos \Omega) + \cos \Omega$$

If you decide to program this operation on your computer, the following numerical example may help in debugging your program. For the rotation

$$\mathrm{ROT}[\mathbf{E}, \Omega] = \mathrm{ROT}[(-37°, 312°), 65°]$$

the values of the elements of the matrix are

$$\mathbf{R} = \begin{bmatrix} 0.588 & 0.362 & -0.724 \\ -0.729 & 0.626 & -0.278 \\ 0.352 & 0.691 & 0.632 \end{bmatrix}$$

The point $\mathbf{A} = (20°, 130°)$ has the Cartesian components $(A_x, A_y, A_z) = (-0.604, 0.720, 0.342)$. Performing the matrix multiplication yields the Cartesian components $(A_x', A_y', A_z') = (-0.342, 0.796, 0.500)$, which when converted to spherical coordinates is (30.0°, 113.2°).

(continued)

Box 7-3. *(continued)*

The rotation ROT[**E**, $-\Omega$] may be regarded as the negative of ROT[**E**, Ω]. These two rotations done in succession leave a point at the coordinates from which it started. (A good check on your computer program is to run the two rotations back-to-back and see if you end up at the point you started with). If the matrix for the operation ROT[**E**, Ω] is the matrix **R** as previously defined, then for the operation ROT[**E**, $-\Omega$] the corresponding matrix is the inverse $\mathbf{R}^{-1}$ of the rotation matrix **R**. If you've taken linear algebra, you'll recognize that the inverse $\mathbf{R}^{-1}$ of a rotation (orthonormal) matrix is the transpose $\mathbf{R}^T$ of the matrix:

$$\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}} = \begin{bmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{bmatrix}$$

where the elements R_{11}, R_{12}, R_{13}, etc. are as defined for the matrix **R**. Upon comparing the matrices **R** and $\mathbf{R}^{\mathrm{T}}$, we find that the first row of $\mathbf{R}^{\mathrm{T}}$ is the first column of **R**, the second row of $\mathbf{R}^{\mathrm{T}}$ is the second column of **R**, and the third row of $\mathbf{R}^{\mathrm{T}}$ is the third column of **R**. This has the effect of exchanging R_{13} with R_{31}, R_{12} with R_{21}, and R_{23} with R_{32}. If you examine the equations defining R_{13} and R_{31}, you will see that changing the sign of Ω is equivalent to exchanging these two elements of **R**, and likewise for R_{12}, R_{21}, and R_{23}, R_{32}.

A produced before 30 Ma remain concentric about ${}_A\mathbf{E}_B$ as they and ${}_A\mathbf{E}_B$ rotate together about the new pole **F**. After rotation about **F** has continued for a while, we can recognize that the pole has jumped to a new position by noting that the older transforms on the two sides of the ridge are not concentric about the same pole. Two Euler poles, ${}_A\mathbf{E}_B$ and ${}_B\mathbf{E}_A$, are needed to fit the transforms. Let's say that the total amount of rotation about **F** from 30 Ma to the present can

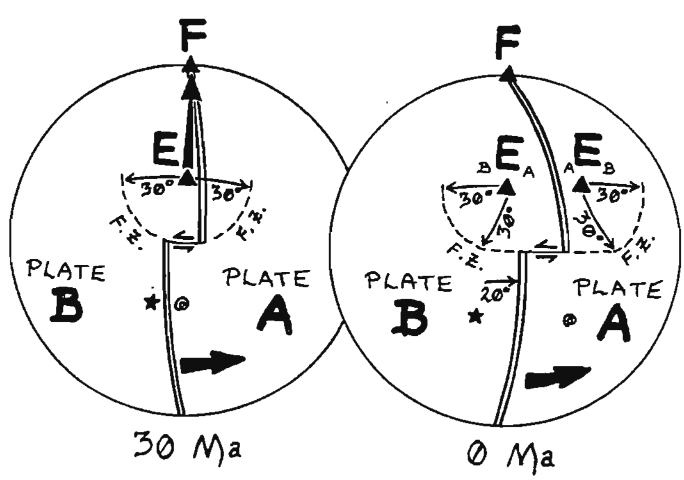
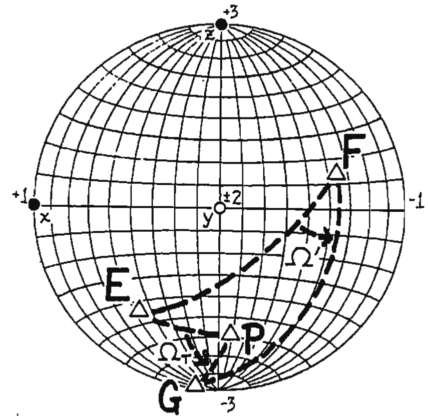


Figure 7-2.
The Euler pole jumps from **E** to **F** at 30 Ma. Prior to that, rotation about **E** produced the arcuate fracture zone shown to the left with a radius of 30°. After the pole jumps to **F** the new fracture zones have a radius of 90°. In the figure all of the points on plate B remain fixed whereas the pole ${}_A\mathbf{E}_B$ and all of the other points on plate A rotate 40° about pole **F**. The active spreading center rotates 20° about pole **F** during the time 30 Ma to the present.

Box 7-4. *(continued)*



- Find **G**

G can be found by rotating **E** by the angle Ω' around **F**. In this example, Ω' is positive and the rotation is counterclockwise.

- Measure angle Ω_T

Point **E** on plate B does not move during the first rotation, ROT[**E**, Ω]. During the second rotation, ROT[**F**, Ω'], **E** moves to **G**. Since any point on plate B can be used to determine the total rotation and since **E** is on plate B, the angle Ω_T, which is the angle between the great circles <**P**, **E**> and <**P**, **G**>, is the combined angle of rotation. Ω_T is measured in the direction going from the great circle <**P**, **E**> to <**P**, **G**> and is positive if this angle is measured in the counterclockwise direction. (If you've forgotten how to find the angle between two great circles, review Box 3-5).

Note that Ω_T remains the same if you reverse the sequence of rotations, that is

$$\text{if ROT}[\mathbf{E}, \Omega] + \text{ROT}[\mathbf{F}, \Omega'] = \text{ROT}[\mathbf{P}, \Omega_T]$$
$$\text{then ROT}[\mathbf{F}, \Omega'] + \text{ROT}[\mathbf{E}, \Omega] = \text{ROT}[\mathbf{Q}, \Omega_T]$$

Note also that finding **Q** is unnecessary in finding ROT[**P**, Ω_T] = ROT[**E**, Ω] + ROT[**F**, Ω'], and is added only to explain what is going on.

Box 7-5. How to Add Rotations Using Matrix Multiplication.

Let's first rotate the point on the globe described by the vector **A** a new position **A′** using the rotation matrix **R** so that **A′** = **R A**. Next let's rotate **A′** using the rotation matrix **R′** so that **A″** = **R′ A′**. How can we find a rotation matrix **T** that will rotate **A** directly to **A″**, such that **A″** = **T A**?

We begin by using the associative property of matrix multiplication. This says that **A″** = **R′ A′** = **R′** (**R A**) = (**R′ R**)**A** = **T A**, or

(continued)

Box 7-5. *(continued)*

$$\mathbf{T} = \mathbf{R'}\,\mathbf{R}$$

the product of the two 3 × 3 matrices.

Matrix multiplication is defined as follows. In element form it is written

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} R'_{11} & R'_{12} & R'_{13} \\ R'_{21} & R'_{22} & R'_{23} \\ R'_{31} & R'_{32} & R'_{33} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

where

$$T_{11} = R'_{11}R_{11} + R'_{12}R_{21} + R'_{13}R_{31}$$
$$T_{12} = R'_{11}R_{12} + R'_{12}R_{22} + R'_{13}R_{32}$$
$$T_{13} = R'_{11}R_{13} + R'_{12}R_{23} + R'_{13}R_{33}$$

$$T_{21} = R'_{21}R_{11} + R'_{22}R_{21} + R'_{23}R_{31}$$
$$T_{22} = R'_{21}R_{12} + R'_{22}R_{22} + R'_{23}R_{32}$$
$$T_{23} = R'_{21}R_{13} + R'_{22}R_{23} + R'_{23}R_{33}$$

$$T_{31} = R'_{31}R_{11} + R'_{32}R_{21} + R'_{33}R_{31}$$
$$T_{32} = R'_{31}R_{12} + R'_{32}R_{22} + R'_{33}R_{32}$$
$$T_{33} = R'_{31}R_{13} + R'_{32}R_{23} + R'_{33}R_{33}$$

In index notation this is written:

$$T_{ij} = RR_{i1}R_{1j} + RR_{i2}R_{2j} + RR_{i3}R_{3j} = \sum_k (R'_{ik}R_{kj})$$

Note that T_{ij}, the element of the *i*th row and *j*th column of **T**, is dot product of the *i*th *row* of **R′** and the *j*th *column* of **R**.

Matrix multiplication does not commute. We shouldn't expect it to because it describes adding rotations, which we know doesn't commute. Algebraically, if we define **T′** = **RR′**, we can show that

$$T_{ij} = \sum_k (R'_{ik}R_{kj}) \neq \sum_k (R_{ik}R'_{kj}) = T'_{ij}$$

and therefore

$$\mathbf{T} = \mathbf{R'}\,\mathbf{R} \neq \mathbf{R}\,\mathbf{R'} = \mathbf{T'}$$

It is very important not to commute when adding rotations.

Now that we have found **T** from matrix multiplication, can we find the Euler pole **E** and finite angle Ω such that **T** = ROT[**E**, Ω]? Yes we can. The trick is to use the definition of the rotation elements listed in Box 7-3 to derive

(continued)

Box 7-5. *(continued)*

$$T_{32} - T_{23} = 2E_x \sin \Omega$$
$$T_{13} - T_{31} = 2E_y \sin \Omega$$
$$T_{21} - T_{12} = 2E_z \sin \Omega$$
$$T_{11} + T_{22} + T_{33} - 1 = 2 \cos \Omega$$

from which we can derive

$$\phi_E = \tan^{-1}\left(\frac{T_{13} - T_{31}}{T_{32} - T_{23}}\right)$$

$$\lambda_E = \sin^{-1}\left(\frac{T_{21} - T_{12}}{\sqrt{(T_{32} - T_{23})^2 + (T_{13} - T_{31})^2 + (T_{21} - T_{12})^2}}\right)$$

$$\Omega = \tan^{-1}\left(\frac{\sqrt{(T_{32} - T_{23})^2 + (T_{13} - T_{31})^2 + (T_{21} - T_{12})^2}}{T_{11} + T_{22} + T_{33} - 1}\right)$$

with the range of Ω such that $0° \leq \Omega \leq 180°$. It is very important to be sure that Ω is within this range. In most computer languages, the arctangent function returns a value from $-90°$ to $90°$, which must be converted to the desired range.

history of the formation of major ocean basins. These shifts correspond to small but real changes in the direction of plate motions. The history of spreading between major plates now appears to consist of a sequence of finite rotations about slightly different Euler poles. The time intervals between successive pole jumps appears to vary from several tens of millions of years to several million years.

In summary, during a sequence of finite rotations about different Euler poles, plates move apart along a series of different small circle arcs, whereas in motion about one Euler pole they move apart along a single arc. Using a single summary rotation is mathematically convenient for making reconstructions, but it does not generally give the true path followed by the plates.

Finite Rotations Versus Angular Velocity Vectors

Both angular velocity vectors and finite rotations can be described by a scalar angle and a unit vector **E**. Both unit vectors are commonly called Euler poles. Because of this