

Ge/Ay 132
Problem Set #2 Solutions

1)a) A fit to the values for the fine structure splittings given in the table with the expression $\Delta E = AZ^B$ is shown in figure 1a.

From this fit,

$$A=2.749 \times 10^{-4} \text{ and } B=5.54167$$

b) Given $\lambda = 53.3 \text{ \AA} \Rightarrow \tilde{\nu} = 18857.25 \text{ cm}^{-1}$,
and plugging into $\Delta E = 2.749 \times 10^{-4} * Z^{5.54167}$,

$$\Rightarrow Z=26$$

So the possible source for this line is the transition between states ${}^2P_{3/2}^{\circ}$ and ${}^2P_{1/2}^{\circ}$ for ground level Fe XIV which is forbidden by electric dipole and pure LS coupling magnetic dipole.

c) In thermal equilibrium, Saha's equation gives

$$\frac{n(\text{Fe XIV}) n(e)}{n(\text{Fe XIII})} = \frac{2 g_{\text{XIV}}}{g_{\text{XIII}}} \frac{[2 \pi M_e k T]}{h^2}^{3/2} * \exp(-I_P(\text{XIII})/kt) \quad (1)$$

and

$$\frac{n(\text{Fe XV}) n(e)}{n(\text{Fe XIV})} = \frac{2 g_{\text{XV}}}{g_{\text{XIV}}} \frac{[2 \pi M_e k T]}{h^2}^{3/2} * \exp(-I_P(\text{XIV})/kt) \quad (2)$$

i.e. for Fe XIV to be the dominant ion, we must have

$$n(\text{Fe XIV}) / n(\text{Fe XIII}) > 1 \quad \text{and} \quad n(\text{Fe XV}) / n(\text{Fe XIV}) < 1$$

So from (1) we have

$$n(e) > \frac{2 g_{\text{XIV}}}{g_{\text{XIII}}} \frac{[2 \pi M_e k T]}{h^2}^{3/2} * \exp(-I_P(\text{XIII})/kt) \quad (3)$$

from (2) we have

$$n(e) > \frac{2 g_{\text{XV}}}{g_{\text{XIV}}} \frac{[2 \pi M_e k T]}{h^2}^{3/2} * \exp(-I_P(\text{XIV})/kt) \quad (4)$$

combine (3) and (4) and we have:

$$\frac{2 g_{XIV}}{g_{XIII}} \frac{[2 \pi M_e k T]^3}{h^2} \exp(-I_P(XIII)/kt) > \frac{2 g_{XV}}{g_{XIV}} \frac{[2 \pi M_e k T]^3}{h^2} * \exp(-I_P(XIV)/kt)$$

$$\text{Fe XIII ground state } ^3P_0 \Rightarrow g_{XIII} = 1$$

$$\text{Fe XIV ground state } ^2P_{1/2}^o \Rightarrow g_{XIV} = 2$$

$$\text{Fe XV ground state } ^1S_0 \Rightarrow g_{XV} = 1$$

$$n(e) = 1.2 \times 10^{10} \text{ cm}^{-3} \quad \text{IP (XIII)} = 361 \text{ eV} \quad \text{IP(XIV)} = 392 \text{ eV}$$

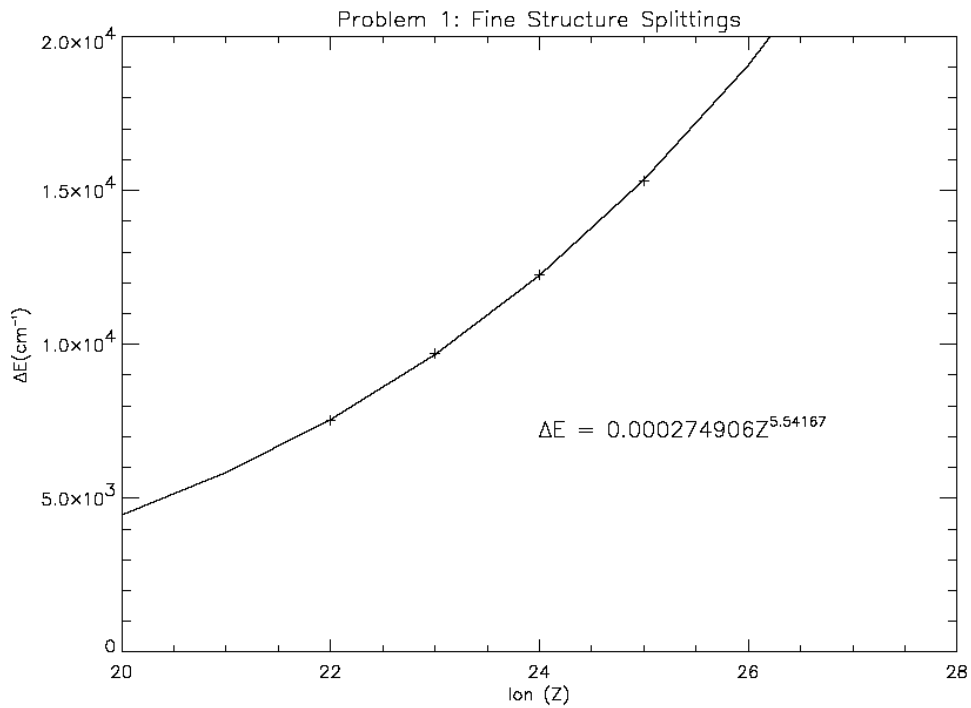
So we have

$$1.24 \times 10^{-6} * \exp(4189206/T) < T^{3/2} < 1.24 \times 10^{-6} * \exp(4548993/T)$$

And the temperature should be

$$1.34 \times 10^5 \text{ K} < T < 1.45 \times 10^5 \text{ K}$$

Figure 1 a:



2)a) Photoionization rate:

$$\begin{aligned}\Gamma_{PI} &= \int_{\nu_0}^{\infty} (J_{\nu}/h\nu) \sigma_{\nu} d\nu \\ &= \int_{\nu_0}^{\infty} 10^{-20} (\nu/\nu_0)^{-1} * 1/h\nu * 6.3 \times 10^{-18} * (\nu/\nu_0)^{-3} d\nu \\ &= 2.38 \times 10^{-12} \text{ s}^{-1}\end{aligned}$$

b) assume photoionization is balanced by radiative recombination, then

$$n(\text{H}) \Gamma_{PI} = \alpha_{\text{H}} * n(\text{H}^+) n(\text{e})$$

and $n(\text{H}^+) = n(\text{e})$

$$n^2(\text{H}^+) \alpha_{\text{H}} = \Gamma_{PI} (n - n(\text{H}^+))$$

$$n(\text{H}^+) = \frac{1}{2} \left\{ \left[\left(\frac{\Gamma_{PI}^2}{\alpha_{\text{H}}} \right) + 4n \left(\frac{\Gamma_{PI}^2}{\alpha_{\text{H}}} \right) \right]^{-1/2} - \left(\frac{\Gamma_{PI}}{\alpha_{\text{H}}} \right) \right\}$$

here $T = 8 \times 10^3 \text{ K}$

$$\alpha_{\text{H}} = 1.8 \times 10^{-10} T^{-0.7} = 3.52 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1}$$

i) for $n = 0.005 \text{ cm}^{-3}$

$$n(\text{H}^+) = \frac{1}{2} \left\{ \left[\left(\frac{2.38 \times 10^{-12}}{3.52 \times 10^{-13}} \right)^2 + 4 * 0.005 * \left(\frac{2.38 \times 10^{-12}}{3.52 \times 10^{-13}} \right) \right]^{-1/2} - \left(\frac{2.38 \times 10^{-12}}{3.52 \times 10^{-13}} \right) \right\}$$

$$n(\text{H}^+) = 0.004996$$

So $x = n(\text{H}^+) / n = 99.93 \%$

ii) for $n = 1.0 \text{ cm}^{-3}$

$$n(\text{H}^+) = 0.88434$$

$$x = n(\text{H}^+) / n = 88.43 \%$$

iii) for $n = 200.0 \text{ cm}^{-3}$

$$n(\text{H}^+) = 33.5477$$

$$x = n(\text{H}^+) / n = 16.77 \%$$

c) In thermal equilibrium, Saha's equation gives

$$\frac{n(\text{H}^+) n(e)}{n(\text{H})} = \frac{2 g_{\text{H}^+}}{g_{\text{H}}} \frac{(2 \pi M_e k T)^{3/2}}{h^2} * \exp(-I_p / kt)$$

⇒

$$\frac{n^2(\text{H}^+)}{n(\text{H})} = \frac{2 g_{\text{H}^+}}{g_{\text{H}}} \frac{(2 \pi M_e k T)^{3/2}}{h^2} * \exp(-I_p / kt)$$

Since $n(\text{H}^+) = x * n$, $n(\text{H}) = n(1 - x)$

and $g = 2J + 1 \Rightarrow g_{\text{H}^+} = 1$ and $g_{\text{H}} = 2$

So

$$\frac{x^2 n}{1-x} * \exp(-I_p / kt) = \frac{2 g_{\text{H}^+}}{g_{\text{H}}} \frac{(2 \pi M_e k T)^{3/2}}{h^2}$$

$x = 99.33 \%$, $n = 0.005 \Rightarrow T = 3450 \text{ K}$

$x = 88.43 \%$, $n = 1.0 \Rightarrow T = 3450 \text{ K}$

$x = 16.77 \%$, $n = 200 \Rightarrow T = 3450 \text{ K}$

For the three cases above, the temperatures should all be around 3450 K to give these various degrees of ionization.

3) He I at 3888.65 Å

a) "Excited long-lived" lower state implies that decay is forbidden. This suggests that the lower level is 2^3S or 2^3P . The large oscillator strength implies that the transition from lower level to upper level is allowed, hence a triplet-triplet transition, $\Delta L = \pm 1$.

b) In the optically thin limit, the equivalent width of this line is:

$$W_\lambda = (\lambda^2 / c) W_\nu = (\lambda^2 / c) * (\pi e^2 / m_e c) N_e f_{ul} = (\lambda^2 N_e f_{ul}) / 1.13 \times 10^{20}$$

$$\lambda = 3888.65 \text{ \AA} \quad f_{ul} = 0.06446 \quad N_e = 3 \times 10^{10} \text{ cm}^{-2}$$

So,

$$W_\lambda = ((3888.65)^2 * (3 \times 10^{10}) * 0.06446) / 1.13 \times 10^{20}$$
$$= 0.000259 \text{ \AA}$$

for $N_e = 3 \times 10^{10} \text{ cm}^{-2}$ $W_\lambda = 0.000259 \text{ \AA}$

It fits the linear part of the curve of growth.

Use $N_e = 10^{12} \text{ cm}^{-2}$ $W_\lambda = 0.0086 \text{ \AA}$

compare to the data in the table, the error is 1.29%

Use $N_e = 3 \times 10^{13} \text{ cm}^{-2}$ $W_\lambda = 0.259 \text{ \AA}$

compare to the data in the table, the error is 37%

c) See attached Figure 3c for $\log W_\lambda$ vs. $\log N$ plot.

d) Using the figure plotted in c)

for $N_e = 3.31 \times 10^{13} \text{ cm}^{-2}$ $W_\lambda = 0.198 \text{ \AA}$

if W_λ is uncertain by 15 %, estimating from the figure, N_e is uncertain by 28 %.

for $N_e = 1.29 \times 10^{13} \text{ cm}^{-2}$ $W_\lambda = 0.098 \text{ \AA}$

if W_λ is uncertain by 15 %, estimating from the figure, N_e is uncertain by 7 %.

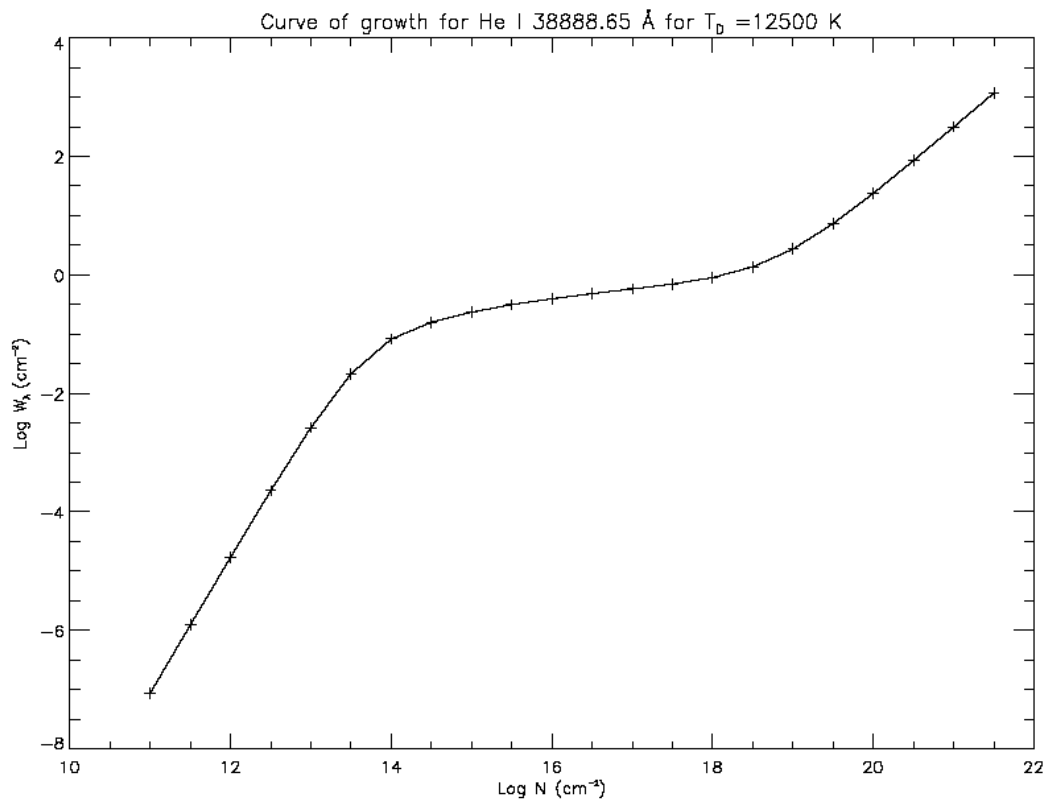
for $N_e = 7.74 \times 10^{12} \text{ cm}^{-2}$ $W_\lambda = 0.050 \text{ \AA}$

if W_λ is uncertain by 15 %, estimating from the figure, N_e is uncertain by 4.3 %.

e) When $W_\lambda = 0.8 \text{ \AA}$, corresponding $N_e = 1.47 \times 10^{17} \text{ cm}^{-2}$

if W_λ is uncertain by 15 %, estimating from the figure, N_e is uncertain by 90 %.

Figure 3c:



4) Consider the sulfur ion S^+ with $3s^2 3p^3$:

The corresponding terms are 4S , 2D and 2P

a) The critical density for level i is:

$$n_c(i) = \frac{\sum_{j < i} A_{ij}}{\sum_{j \neq i} q_{ij}}$$

$$\text{when } j < i, q_{ij} = q_{ul} = (8.629 \times 10^{-6} * \Omega) / (T^{1/2} * g_u), \quad \text{where } g_u = 2J_u + 1$$

$$\text{when } j > i, q_{ij} = (8.629 \times 10^{-6} * \Omega) / (T^{1/2} * g_L), \quad \text{where } g_L = 2J_L + 1$$

for $^4S_{3/2}^0$,

$$\sum_{j < i} A_{ij} = 0 \quad \text{and} \quad n_c = 0$$

for $^2D_{3/2}^0$,

$$\sum_{j < i} A_{ij} = 8.82 \times 10^{-4} \text{ s}^{-1}$$

$$\begin{aligned} \sum_{j \neq i} q_{ij} &= q(^2D_{3/2} \rightarrow ^2D_{5/2}) + q(^2D_{3/2} \rightarrow ^2P_{3/2}) + q(^2D_{3/2} \rightarrow ^2P_{1/2}) + q(^2D_{3/2} \rightarrow ^4S_{3/2}) \\ &= 8.629 \times 10^{-6} / [(8 \times 10^3)^{1/2} (2 * 3/2 + 1)] * [7.59 * \exp -(14884.8 - 14853.0)/kt \\ &\quad + 3.38 * \exp -(26571.8 - 14853.0)/kt + 1.52 * \exp -(24524.4 - 14853.0)/kt \\ &\quad + 8.629 \times 10^{-6} / [(8 \times 10^3)^{1/2} (2 * 3/2 + 1)] * 2.79 \end{aligned}$$

So for $^2D_{3/2}^0$, $n_c = 4.3 \times 10^3 \text{ cm}^{-3}$

For $^2D_{5/2}^0$,

$$\sum_{j < i} A_{ij} = 2.60335 \times 10^{-4} \text{ s}^{-1}$$

$$\sum_{j \neq i} q_{ij} = \sum q_{lu} + \sum q_{ul}$$

the allowed transitions are

$$l \rightarrow u : {}^2D_{5/2} \rightarrow {}^2P_{3/2}, {}^2P_{1/2}$$

$$u \rightarrow l : {}^2D_{5/2} \rightarrow {}^2D_{3/2}, {}^4S_{3/2}$$

$$\sum_{j \neq l} q_{ij} = 2.08 \times 10^{-7} \text{ cm}^3/\text{s}$$

$$n_c = 1.251 \times 10^3 \text{ cm}^{-3}$$

For ${}^2P_{1/2}^o$,

$$\sum_{j < i} A_{ij} = 0.3315 \text{ s}^{-1}$$

$$\sum_{j \neq l} q_{ij} = \sum q_{lu} + \sum q_{ul}$$

the allowed transitions are

$$l \rightarrow u : {}^2P_{1/2} \rightarrow {}^2P_{3/2}$$

$$u \rightarrow l : {}^2P_{1/2} \rightarrow {}^2D_{5/2}, {}^2D_{3/2}, {}^4S_{3/2}$$

$$\sum_{j \neq l} q_{ij} = 3.43 \times 10^{-7} \text{ cm}^3/\text{s}$$

$$n_c = 9.66 \times 10^5 \text{ cm}^{-3}$$

For ${}^2P_{3/2}^o$,

$$\sum_{j < i} A_{ij} = 0.537 \text{ s}^{-1}$$

$$\sum_{j \neq l} q_{ij} = \sum q_{ul}$$

the allowed transitions are

$$u \rightarrow l : {}^2P_{3/2} \rightarrow {}^2P_{1/2}, {}^2D_{5/2}, {}^2D_{3/2}, {}^4S_{3/2}$$

$$\sum_{j \neq l} q_{ij} = 2.89 \times 10^{-7} \text{ cm}^3/\text{s}$$

$$n_c = 1.86 \times 10^6 \text{ cm}^{-3}$$

b) For 4S ,

$$\begin{aligned}
 n(^4S) n(e) & \left[\frac{q(^4S \rightarrow ^2D)}{(1)} + \frac{q(^4S \rightarrow ^2P)}{(2)} \right] \\
 & = n(^2D) \frac{q(^2D \rightarrow ^4S) n(e)}{(3)} + \frac{A(^2D \rightarrow ^4S)}{(4)} \\
 & + n(^2P) \left[\frac{q(^2P \rightarrow ^4S) n(e)}{(5)} + \frac{A(^2P \rightarrow ^4S)}{(5)} \right]
 \end{aligned}$$

We assume the populations of the 3 levels differ by a lot due to almost 10 cm^{-1} difference in ΔE . (We can check this assumption later.)

Since 2P has high energy, we can ignore (5)

Compare with (3) and (4):

use $n(e) = 100 \text{ cm}^{-3}$ $T = 8 \times 10^3 \text{ K}$

$$\begin{aligned}
 q(^2D \rightarrow ^4S) & = 8.629 \times 10^{-6} * \Omega_M / [T^{1/2} (2*s_u+1) * 2(u+1)] \\
 & = 8.629 \times 10^{-6} * (4.19+2.79) / [(8 \times 10^3)^{1/2} (2*1/2+1) * 2*(2+1)] \\
 & = 6.7 \times 10^{-8} \text{ cm}^{-3}/\text{s}
 \end{aligned}$$

So

$$(3) = 6.7 \times 10^{-8} * 100 = 6.7 \times 10^{-6} \text{ s}^{-1}$$

$$(4) = A(^2D \rightarrow ^4S) = (2.60 + 8.82) \times 10^{-4} = 1.14 \times 10^{-3} \text{ s}^{-1}$$

(4) \ll (3), so term (3) can be ignored.

Compare with (1) and (2):

use $n(e) = 100 \text{ cm}^{-3}$ $T = 8 \times 10^3 \text{ K}$

$$\begin{aligned}
 (1) = q(^4S \rightarrow ^2D) & = 8.629 \times 10^{-6} * \Omega_M / [T^{1/2} (2*s_l+1) * 2(l+1)] * \exp - \Delta E / kt \\
 & = 8.629 \times 10^{-6} * (4.19+2.79) / [(8 \times 10^3)^{1/2} (2*3/2+1) * 2*(0+1)] \\
 & \quad * \exp -(14870-1.44)/(8 \times 10^3) \\
 & = 1.2 \times 10^{-8} \text{ cm}^{-3}/\text{s}
 \end{aligned}$$

$$\begin{aligned}
(2) = q ({}^4\text{S} \rightarrow {}^2\text{P}) &= 8.629 \times 10^{-6} * (1.52 + 0.759) / [(8 \times 10^3)^{1/2} (2 * 3/2 + 1) * 2 * (0 + 1)] \\
&* \exp -(24550 - 1.44) / (8 \times 10^3) \\
&= 6.6 \times 10^{-10} \text{ cm}^{-3}/\text{s}
\end{aligned}$$

So (1) >> (2), so term (2) can be ignored.

In summary, for ${}^4\text{S}$ the dominant term is:

$$n({}^4\text{S}) n(e) q ({}^4\text{S} \rightarrow {}^2\text{D}) = n({}^2\text{D}) n(e) A ({}^2\text{D} \rightarrow {}^4\text{S})$$

For ${}^2\text{D}$:

$$\begin{aligned}
n({}^2\text{D}) [\underbrace{A ({}^2\text{D} \rightarrow {}^4\text{S})}_{(1)} + \underbrace{n(e) \{ q ({}^2\text{D} \rightarrow {}^4\text{S}) + q ({}^2\text{D} \rightarrow {}^2\text{P}) \}}_{(2) \quad (3)}] \\
= n({}^4\text{S}) n(e) [q ({}^4\text{S} \rightarrow {}^2\text{D})] + \underbrace{n({}^2\text{P}) [n(e) q ({}^2\text{P} \rightarrow {}^{\text{S}}\text{D}) + A ({}^2\text{P} \rightarrow {}^{\text{S}}\text{D})]}_{(4)}
\end{aligned}$$

Since $n({}^2\text{P})$ tends to be much smaller than $n({}^2\text{D})$ and $n({}^4\text{S})$, (4) can be ignored.

Compare (2) and (3):

$$\begin{aligned}
(2) = q ({}^2\text{D} \rightarrow {}^4\text{S}) &= 6.7 \times 10^{-8} \\
(3) = q ({}^2\text{D} \rightarrow {}^4\text{S}) &= 8.629 \times 10^{-6} * 4.56 / [(8 \times 10^3)^{1/2} (2 * 1/2 + 1) * 2 * (1/2 + 1)] \\
&* \exp -(9680 * 1.44) / (8 \times 10^3) \\
&= 7.7 \times 10^{-9} \text{ cm}^{-3}/\text{s}
\end{aligned}$$

So (2) > (3)

but, $(1) = A ({}^2\text{D} \rightarrow {}^4\text{S}) = 1.14 \times 10^{-3} \gg n(e) ((2) + (3))$

For ${}^2\text{D}$ in summary the dominant term is the same as in ${}^4\text{S}$:

$$n({}^4\text{S}) n(e) q ({}^4\text{S} \rightarrow {}^2\text{D}) = n({}^2\text{D}) A ({}^2\text{D} \rightarrow {}^4\text{S})$$

For ${}^2\text{P}$:

$$n({}^2\text{P}) \left[\frac{A({}^2\text{P} \rightarrow {}^2\text{D})}{(1)} + \frac{A({}^2\text{P} \rightarrow {}^4\text{S})}{(2)} + n(e) \left[\frac{q({}^2\text{P} \rightarrow {}^2\text{D})}{(3)} + \frac{q({}^2\text{P} \rightarrow {}^4\text{S})}{(4)} \right] \right]$$

$$= n(e) \left[n({}^2\text{D}) q_m({}^2\text{D} \rightarrow {}^2\text{P}) + n({}^4\text{S}) q({}^4\text{S} \rightarrow {}^2\text{P}) \right]$$

$$(1) = A({}^2\text{P} \rightarrow {}^2\text{D}) = 0.197 + 0.133 + 0.0779 + 0.163 = 0.5529 \text{ s}^{-1}$$

$$(2) = A({}^2\text{P} \rightarrow {}^4\text{S}) = 0.225 + 0.0906 = 0.3156 \text{ s}^{-1}$$

$$(3) = q({}^2\text{P} \rightarrow {}^2\text{D}) = 8.629 \times 10^{-6} / (8 \times 10^3)^{1/2}$$

$$(4) = q({}^2\text{P} \rightarrow {}^4\text{S}) = 8.629 \times 10^{-6} / (8 \times 10^3)^{1/2} * (1.52 + 0.759) / [(2 * 1/2 + 1) * (2(1 + 1))]$$

$$= 3.66 \times 10^{-8}$$

So, $n(e) \left((3) + (4) \right) \sim 10^{-6} \ll (1) \text{ and } (2)$

For ${}^2\text{P}$ in summary, the dominant terms are:

$$n({}^2\text{P}) \left[A({}^2\text{P} \rightarrow {}^2\text{D}) + A({}^2\text{P} \rightarrow {}^4\text{S}) \right] = n(e) \left[n({}^2\text{D}) q_m({}^2\text{D} \rightarrow {}^2\text{P}) + n({}^4\text{S}) q({}^4\text{S} \rightarrow {}^2\text{P}) \right]$$

c) From the term ${}^2\text{D}$ or ${}^4\text{S}$ we have :

$$n({}^4\text{S}) n(e) q({}^4\text{S} \rightarrow {}^2\text{D}) = n({}^2\text{D}) n(e) A({}^2\text{D} \rightarrow {}^4\text{S})$$

$$\frac{n({}^4\text{S})}{n({}^2\text{D})} = \frac{A({}^2\text{D} \rightarrow {}^4\text{S})}{n(e) q({}^4\text{S} \rightarrow {}^2\text{D})}$$

$$= \frac{1.14 \times 10^{-3} \text{ s}^{-1}}{100 * 1.2 \times 10^{-8}}$$

$$= 950$$

Considering the equation for ${}^2\text{P}$ divided by $n({}^2\text{D})$ we have:

$$\left(\frac{n({}^2\text{P})}{n({}^2\text{D})} \right) \left[A({}^2\text{P} \rightarrow {}^2\text{D}) + A({}^2\text{P} \rightarrow {}^4\text{S}) \right]$$

$$= n(e) \left[q_m({}^2\text{D} \rightarrow {}^2\text{P}) + \left(\frac{n({}^4\text{S})}{n({}^2\text{D})} \right) q({}^4\text{S} \rightarrow {}^2\text{P}) \right]$$

So

$$\begin{aligned} \frac{n(^2P)}{n(^2D)} &= \frac{n(e) [q_m(^2D \rightarrow ^2P) + (n(^4S)/n(^2D)) q(^4S \rightarrow ^2P)]}{A(^2P \rightarrow ^2D) + A(^2P \rightarrow ^4S)} \\ &= \frac{100 * [7.7 \times 10^{-9} \text{ s}^{-1} * 950 * 6.6 \times 10^{-10}]}{0.5529 + 0.3156} \\ &= 7.3 \times 10^{-5} \end{aligned}$$

Which probes our assumption, i.e. $n(^2P)$ is much less than $n(^2D)$ and $n(^4S)$

In summary, the relative population of the 3 levels are

$$n(^4S) : n(^2D) : n(^2P) = 1.0 : 1.05 \times 10^{-3} : 7.3 \times 10^{-8}$$

d) In the low density limit, $n(e) \rightarrow 0$, every collisional excitation is followed by the emission of a photon, so the intensity ratio is determined by the relative excitation rate of $^2D_{3/2}$ and $^2D_{5/2}$, which is proportional to their statistical weights $g = 2J + 1$, so

$$\frac{I(^2D_{5/2} \rightarrow ^4S_{3/2})}{I(^2D_{3/2} \rightarrow ^4S_{3/2})} = \frac{2 * 5/2 + 1}{2 * 3/2 + 1} = \frac{6}{4} = 1.5$$

At high density, $n(e) \rightarrow \infty$ collisional excitations and de-excitations dominate and setup a Boltzman population ratio for the two levels and . So the relative population of the two levels is

$$\frac{I(^2D_{5/2} \rightarrow ^4S_{3/2})}{I(^2D_{3/2} \rightarrow ^4S_{3/2})} = \frac{n(^2D_{5/2}) A(^2D_{5/2} \rightarrow ^4S_{3/2})}{n(^2D_{3/2}) A(^2D_{3/2} \rightarrow ^4S_{3/2})} = \frac{6 * 2.6 \times 10^{-4}}{4 * 8.82 \times 10^{-4}} = 0.44$$

In the neighborhood of the critical density ($4.3 \times 10^3 \text{ cm}^{-3}$ for $^2D_{3/2}$ and $1.25 \times 10^3 \text{ cm}^{-3}$ for $^2D_{5/2}$), the intensity ratio is most sensitive to density.