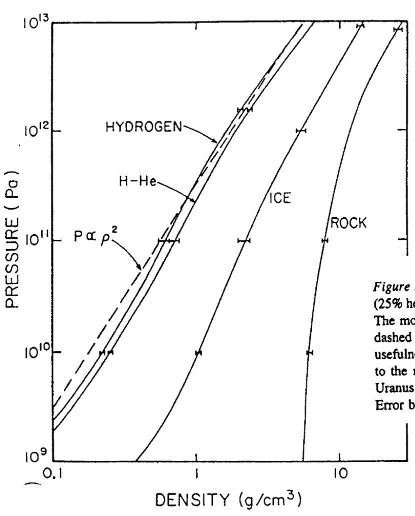
9 Generic Planetary Models

9.1 Nature of the Equations of State

At sufficiently low pressures in solids, the density of a material is little affected by pressure and $M \propto \rho R^3$ with $\rho \sim$ constant. You then expect that the radius of an object will vary as the cube-root of mass:

$$R \propto M^{1/3} \tag{9.1}$$

This makes sense for the low-mass behavior if the material has a finite density as the pressure goes to zero. Recall that the low-mass behavior requires $P \ll K$, where K is the bulk modulus. Since $P \propto R^2$ from hydrostatic equilibrium and $\rho \sim \rho_o(1+P/K)$ where ρ_o is the zero-pressure density (chapter 3), the mean density of a planet will deviate quadratically in R from the zero pressure density as the radius of the planet increases. However, this makes no sense for any planet where hydrogen and helium are significant, because these materials can expand without bound as the pressure is lowered at finite temperature. (Equivalently, P/K=1 in ideal gases, even if P is very small). Even materials that expand to a finite density as $P \rightarrow 0$ will begin to deviate from the simple cube root behavior, at sufficiently high pressure.



Stevenson
Ann. Rev. Earth & Planet. Sci. (1982)

Figure 2 Typical isentropes relevant to the giant planets. The hydrogen and hydrogen-helium (25% helium by mass) isentropes pass through $T = 170$ K at $P = 10^5$ Pa (appropriate to Jupiter). The molecular-metallic hydrogen transition has been smoothed but may be discontinuous. The dashed line is the best fitting curve of the form $P = K\rho^2$ for pure hydrogen, indicating the usefulness of this simple analytic approximation. The isentropes of "ice" and "rock" correspond to the mixtures used by Hubbard & MacFarlane (1980) and are chosen to be appropriate for Uranus (i.e. $T \simeq 2000$ K at $P = 2 \times 10^{10}$ Pa for ice, $T \simeq 7000$ K at $P = 6 \times 10^{11}$ Pa for rock). Error bars indicate the uncertainties.

Figure 9.1

Temperature (e.g. whether the body is adiabatic) also has a significant effect.

9.2 Polytropes

Notice that the figure above shows equations of state that are very roughly straight lines on a log-log plot. This suggest a power law relationship. A polytrope is defined to be an equation of state in such a form: $P \propto \rho^n$.* This will have relevance for materials that expand without limit as $P \rightarrow 0$, or bodies that are so massive that the mean density is much larger than the zero pressure density. Since the central pressure according to hydrostatic equilibrium must scale as $\langle\rho\rangle gR \sim (M/R^3)(GM/R^2)R \sim GM^2/R^4$ and the density scales as M/R^3, we immediately have

$$\frac{M^2}{R^4} \propto \left[\frac{M}{R^3}\right]^n \quad \Rightarrow \; R \propto M^{\frac{2-n}{4-3n}} \tag{9.2}$$

Notice three things:

(i) We recover the result $R \propto M^{1/3}$ in the limit $n \rightarrow \infty$; this makes sense since that limit is incompressible material (infinitesimal density change gives large pressure change).

(ii) We get $R \propto M^{-1/3}$ when n=5/3. Recall that this is the case for an ideal Fermi gas. It is therefore applicable to superJupiters and white dwarfs. At sufficiently high mass (but still non-relativistic) it applies to all materials irrespective of atomic mass. *At sufficiently high mass, all degenerate bodies become smaller as you add mass to them.*

(iii) We get R independent of mass when n=2; this is approximately relevant to Jupiter and Saturn.

(iv) We get no sensible result when n=4/3. Although it is not obvious without further analysis, this turns out to be a hint of instability and the origin of the Chandrasekhar limit to massive stars, since n=4/3 corresponds to the ideal Fermi gas limit for relativistic electrons. (It also corresponds to the limit where radiation pressure dominates).

There is a large literature on polytropes and you can find out all about them in Chandrasekhar's book on stellar structure. In general, the equation of hydrostatic equilibrium does not have an analytical solution, but n=2 is a special case since it leads to a linear differential equation. Let's derive this, since it is of practical use. We assume $P = K\rho^2$:

$$\begin{aligned} \frac{dP}{dr} &= 2K\rho(r)\frac{d\rho(r)}{dr} = -\rho(r)g)(r) \\ 2K\frac{d\rho(r)}{dr} &= -g(r) = -\frac{G}{r^2}\int_0^r 4\pi\rho(x)x^2dx \end{aligned} \tag{9.3}$$

Multiplying by r^2 and performing another derivative eliminates the integral:

$$\frac{d}{dr}(r^2\frac{d\rho}{dr}) = -k^2r^2\rho; \quad k^2 \equiv \frac{2\pi G}{K} \tag{9.4}$$

*In the traditional astrophysical literature, the equation is written $P \propto \rho^{1+1/n}$ and n is called the polytropic index, but here I will use n to mean simply the power law. Thus n=2 in my notation will be the same as "a polytrope of index unity" in the astrophysical notation.

This can be conveniently be written as a standard differential equation

$$\frac{d^2}{dr^2}(r\rho) = -k^2 r\rho \tag{9.5}$$

for which the solutions for rρ(r) are sin(kr) and cos(kr) and it is convenient to wrote the solution this way:

$$\rho(r) = A\frac{\sin kr}{kr} + B\frac{\cos kr}{kr} \tag{9.6}$$

If this solution applies for all radii including the center (r=0) then finiteness requires that B=0. In that case, A is identified as the central density ρ_c because sinx/x is unity as x→0. In this coreless case, the outer surface of the planet r=R must be the first zero of sin(kr)/kr:

$$kR = \pi \quad \Rightarrow R = \sqrt{\frac{\pi K}{2G}} \tag{9.7}$$

Thus we get an explicit formula for the radius, and it is independent of mass, as promised. For the realistic choice of K=2.1 x 10^{12} cgs (a cosmic hydrogen/helium mixture), this formula gives a radius of 70,300 km. The mean radius of Jupiter is 69,800km. We can also compute the mean density in terms of the central density:

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = 3\int_0^1 \rho_c \frac{\sin \pi x}{\pi x} x^2 dx = \frac{3}{\pi^2}\rho_c \tag{9.8}$$

The inferred central density is $\pi^2/3$ times the mean density, corresponding to 4.38 g/cc and a pressure of 40 Megabars for Jupiter. The radius thus obtained should apply equally well to Saturn, but the observed radius of Saturn is only ~58,000km. *The fact that Saturn is smaller than Jupiter must be because it has heavier constituents, not because it has lower mass.* Detailed models confirm this elementary observation.

A common way of estimating the effect of a uniform admixture of heavier elements is to assume *volume additivity.* This is typically accurate to about a percent or two, sufficiently precise for the purpose here. If the mass fraction of "heavies" is y then we get

$$\frac{1}{\rho(P)} = \frac{1-y}{\sqrt{P/K}} + \frac{y}{\rho_{heavy}} \tag{9.9}$$

where ρ(P) is the density at pressure P; the LHS is just the specific volume and the RHS terms represent the specific volumes of the hydrogen–helium mixture and the heavy component respectively. In the limit where the heavy component is very much more dense than the hydrogen–helium mix and y is not too large, this equation is simply $P = K_{eff}\rho^2$ where $K_{eff} = K(1-y)^2$. This is still of the right functional form for application of eq (9.7) and the radius is accordingly reduced by 1-y. Saturn would require that y~0.2, implying ~15 Earth masses of heavier stuff. The <u>same</u> amount of heavy stuff in Jupiter is a smaller fraction of the total mass ($y_{Jupiter}$~0.05) and permitted by the observed radius! Detailed models allow this, though the uncertainties remain large (see Guillot T., ``Review: Interiors of giant planets inside and outside the Solar System", *Science* **286**, 72-77, 1999; also more recent papers on his web site.) The presence of a core (as distinct from just heavy element enrichment) is unresolved, though likely, especially for Saturn. The most recent unpublished models (using a new equation of state from Burkhard Militzer) suggest a 15 Earth mass core for Jupiter. Determination of the presence of a core cannot be done from radius alone; it requires careful consideration of the gravitational moments (discussed in a later chapter).

Solution 9.6 can still be used in the region external to a core, but A is no longer the central density and B is no longer zero. The coefficient B is then determined by making sure that the equation of hydrostatic equilibrium is correctly satisfied at the core surface (i.e., you have to allow for the gravitational acceleration of the core). The relative magnitudes of A and B then determine the shift in the radius R relative to the value given by equation 9.7 (see problem 9.2)

The simple solution (with B=0) also apples to the equation of state

$$P = K(\rho^2 - \rho_o^2) \tag{9.10}$$

but now the radius satisfies $\sin(kR)/kR = \rho_o/\rho_c$ corresponding to P=0. There is no real material with this equation of state but it is nonetheless of interest since it provides a simple model with which to understand the trade-off between the surface density (which must be ρ_o) and compression at depth in determining the radius–mass relationship. The central density is related to the mass through the equation $M = 4\pi\rho_c\,[\sin(kR) - kR\cos(kR)]/k^3$. This equation and $\sin(kR)/kR = \rho_o/\rho_c$ define a transcendental pair of equations that has no simple solution, but can be evaluated in two limits. In the low mass limit where ρ_o/ρ_c is only slightly smaller than unity, we obtain $M = 4\pi\rho_o R^3 (1 + k^2R^2/15)/3$. In the limit $\rho_o/\rho_c \ll 1$, one obtains $kR = \pi(1-4\pi\rho_o/k^3M)$ provided the correction is not too large, which means that the radius goes down as the mass goes down but the dependence on mass is weak once M is large compared to $4\pi\rho_o/k^3$. These two limiting behavior approximately match at $M \sim 4\pi\rho_o/k^3$, $kR \sim \pi/2$, $\rho_o/\rho_c \sim 0.5$.

9.3 Mass-Radius Relationships

In the first figure (below) we see that hydrogen-helium adiabatic bodies have roughly constant radius (as promised) but actually expanding as they approach low mass ideal gas adiabatic behavior ($P \propto \rho^{1.45}$ for a cosmic H_2 -He mixture). The radius actually declines as you go to still higher masses (the brown dwarf regime) though the effect is modest if (as is usually the case) these bodies are also hotter and thus less close to the degenerate limit given by the solid lines. This figure also shows us that Uranus and Neptune do not have a simple interpretation.

Mass-Radius Relationships

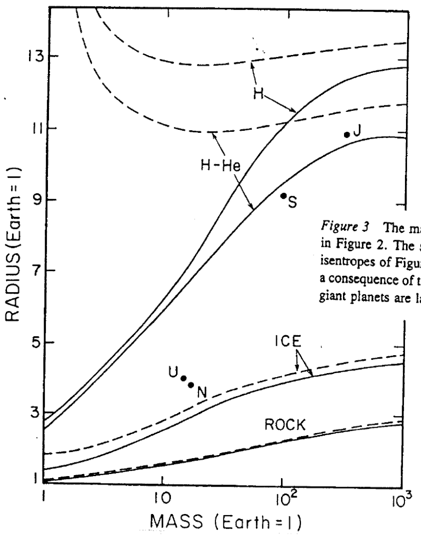


Figure 3 The mass-radius relationship for self-gravitating bodies of the same compositions as in Figure 2. The solid lines are for cold matter ($T = 0$ K); the dashed lines correspond to the isentropes of Figure 2. The insensitivity of radius or mass for hydrogen and hydrogen-helium is a consequence of the approximate validity of $P \propto \rho^2$ (see text for discussion). The positions of the giant planets are labelled by J, S, U, and N.

Stevenson (1982)

Figure 9.2

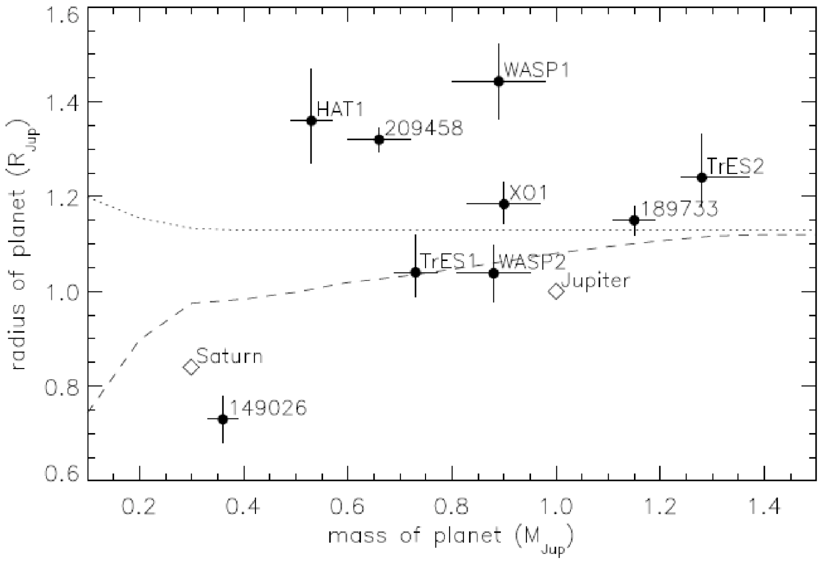


Figure 9.3

In recent years, it has become possible to measure radii of transiting planets of other stars; above is a compilation for those within 300pc (with Jupiter and Saturn included for comparison). The dotted line corresponds to the insolated coreless structural models of Bodenheimer et al. (2003) for an age of 4.5 Gyr and a planetary effective temperature of 1500 K. The dashed line shows their models for the same parameters but including the presence of a 20-M⊕ core of solid material. Insolation alone is clearly insufficient to account for the large radii of three of the planets (HAT-P-1b, WASP-1b, and HD 209458b) and likely a fourth (TrES-2), regardless of whether or not a core is present. Interestingly, the parent stars of these four planets are significantly more massive than those of the planets that are in good agreement with the models (TrES-1, WASP-2b, and HD 189733b), all of which orbit lower-mass, K dwarf stars. Evidently there is some additional source of heating, at least in some bodies, causing them to have a significantly larger radius than the degenerate configuration.

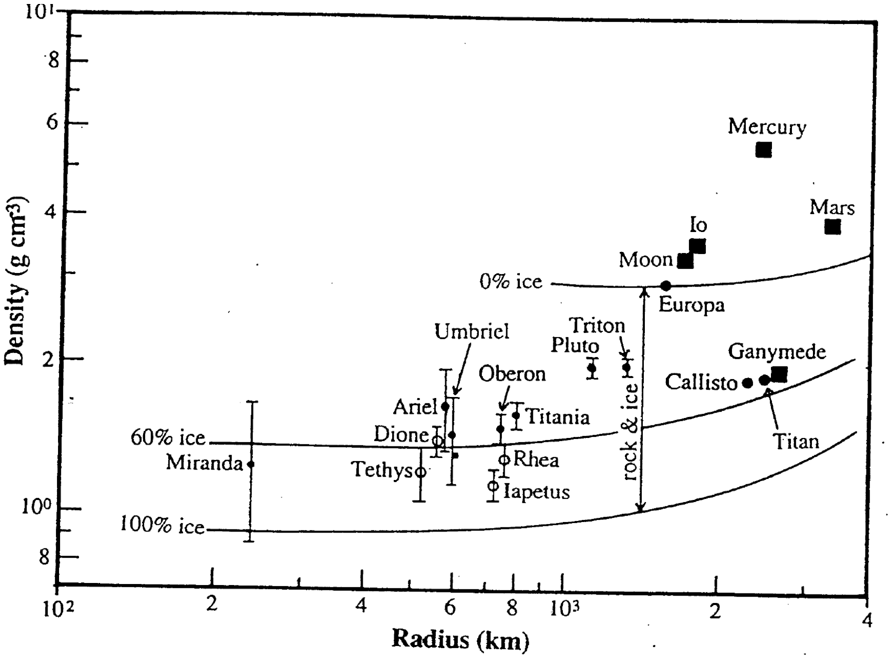


Figure VI.30 Densities of the satellites of the outer planets. A solar-proportion mixture of water ice and rock (60%), with an uncompressed density of 1.3 to 1.5, could be reconciled with many of these data. The Saturnian satellites show a wide spread of densities without any clear radial trend.

Figure 9.4

In the figure above we see the radius–mass relationships for low-mass bodies; primarily the satellites in the outer solar system. Here, one sees that the difference in mean density between Ganymede and (say) Dione is in major part due to ice compression (more precisely, phase changes) rather than intrinsic compositional difference.

9.4 The Virial Theorem

Hydrostatic equilibrium together with the first law of thermodynamics and other thermodynamic relations can enable one to construct an evolution of a planet or star. However, there is an extremely valuable result called the *virial theorem* that enables us to see some of the general behavior of an evolving body. There is no new physics in this theorem, merely a clever use of the existing physics. It assumes slow evolution relative to dynamical timescales, and in the current application this is the same as the hydrostatic assumption. The virial theorem also shows up in other areas of physics (e.g the expected velocity dispersion of a self-gravitating star cluster).

The gravitational energy of a planet can be written

$$E_G = -\int_0^M \frac{Gmdm}{r(m)} \tag{9.11}$$

where we have chosen to use m (the mass inside radius r) as the independent variable. Thus, r is the radius that contains mass m. But we also have the hydrostatic equation, which together with the definition of m(r) leads to the following:

$$\begin{aligned} &\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \\ &dm = 4\pi r^2 \rho dr \\ &\Rightarrow \frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \end{aligned} \tag{9.12}$$

Looking back at the gravitational energy, we see that

$$E_G = \int_0^M 4\pi r^3\, dP = 3\int_0^M \{d[\frac{4}{3}\pi r^3 P] - PdV\} \tag{9.13}$$

However, P.($4\pi r^3$ /3) is zero at both the center and outer surface of the planet, since r=0 and P=0 respectively, at those locations. Consequently, only the PdV terms remains:

$$E_G = -3\int_0^M P\,dV = -3\int_0^M \frac{P}{\rho}dm \tag{9.14}$$

This last result is one of several ways of stating Virial theorem.

In the conventional discussion of stellar structure (*not planets!*), ideal gas formulas apply to a good approximation. The pressure is then nk_BT, where n is the number density of atoms or molecules. Moreover, the thermal energy can be written in the form nk_BT/γ where $\gamma = C_p/C_v - 1$ and is the Gruneisen parameter as previously defined. (Warning: Some texts use γ to represent the ratio of specific heats. Remember also that this formula is only correct for an ideal gas.) It follows that

$$E_{total} = E_G + E_{thermal} = -(3 - \frac{1}{\gamma})\int_0^M P\,dV \tag{9.15}$$

whence it follows immediately that provided $\gamma > 1/3$, the total energy is negative. Atomic hydrogen has $\gamma = 2/3$, molecular hydrogen has $\gamma = 2/5$. Radiation dominated systems can approach $\gamma = 1/3$ and the specialness of this limit is related to the undetermined character of a polytrope when n=4/3 and the resulting instability (collapse or explosion), as discussed at the beginning of this chapter.

The following discussion is only for the usual case where $\gamma > 1/3$. For a body in hydrostatic equilibrium and supported by ideal gas pressure, the integral in 9.15 increases as the temperature goes up since P=nkT and the integral of ndV is constant. But this corresponds ot a decrease in total energy and therefore corresponds to contraction as heat is lost to space. In other words, a star with no other energy sources has "negative specific heat": As it loses energy by radiation to space, it contracts, and heats up internally. This was a major puzzle in the early work on stellar evolution. In reality, the contraction and heating is truncated either by the introduction of a new energy source: thermonuclear ignition (arrival at the main sequence) or the introduction of a new pressure: the onset of degeneracy. Provided ideal gas dominates, $T \sim GM\mu/k_BR$, either during slow contraction or on the main sequence. Here, T is the typical internal temperature and μ is the molecular weight (normally ~ proton mass for a main sequence star).

In a planet, you cannot use the Virial theorem to determine temperature directly, but you can still use it to set up equations for the thermal history, as explained below, and it leads to a very different answer from stars.

It is most useful to consider a perturbation to the planet in which hydrostatic equilibrium is preserved, but there are infinitesimal changes in density and pressure at each mass element. In the following formulas, you should think of δ as labeling the change that comes about because of some evolutionary change at that location. For example, $\delta(1/r)$ means the change in 1/r that comes from the change in the radius that contains mass m. Thus,

$$\delta E_G = -\int_0^M Gm\delta(\frac{1}{r})dm = \int_0^M \frac{Gm}{r^2}\delta(r)dm \tag{9.16}$$

But as before we apply hydrostatic equilibrium (eq. 9.12) to get

$$\begin{aligned}\delta E_G &= -\int_0^M 4\pi r^2 \frac{dP}{dm}\delta(r)dm = -\int_0^M \delta(\frac{4}{3}\pi r^3)\frac{dP}{dm}dm \\ &= -\int_0^M \{d[\delta(\frac{4}{3}\pi r^3)P] - \delta[\frac{d(\frac{4}{3}\pi r^3)}{dm}]Pdm\}\end{aligned} \tag{9.17}$$

However, the first term is zero just as it was previously, and $dm = \rho.d(4\pi r^3/3)$, whence

$$\delta E_G = \int_0^M P.\delta\left(\frac{1}{\rho}\right)dm \tag{9.18}$$

So the change in gravitational energy equals the work done on the sample. (This interpretation only works if you think about constant composition. Obviously you can also lower the energy by moving the denser stuff to higher pressures and the less dense stuff to lower pressures, i.e., δ(1/ρ) is negative where P is high and positive where P is low. The formula is always correct but the interpretation of it depends on the circumstances).

We can now use this result to derive a very important result for the energy output of planets.

9.5 Planet Luminosity

The intrinsic luminosity L of a planet comes from explicit energy sources (e.g. radioactive decay, tidal heating), here labeled Q (per unit mass) but can also come from changes in internal and gravitational energy. When we refer to “intrinsic” luminosity we are excluding the (sometimes dominant) energy arising from external source, especially the absorbed sunlight. Thus,

$$L = \int_0^M Q\,dm - \frac{d(E_G + E_{\text{int}})}{dt} \tag{9.19}$$

Consider a planet that is not differentiating (i.e., not changing the distribution of constituents). Now, planets are degenerate bodies and we can thus conveniently subdivide the internal energy and pressure into zero temperature pieces and finite temperature corrections in the form:

$$\begin{aligned} E_{\text{int}} &= E_0 + C_V T \\ P &= P_0 + \gamma\rho C_V T \\ P_0 &= -\frac{dE_0}{d(\frac{1}{\rho})} \end{aligned} \tag{9.20}$$

(This approximation is not essential to the result but aids the understanding of the result.) So:

$$\frac{d(E_{\text{int}} + E_G)}{dt} = \int_0^M \left[\frac{dE_0}{dt} + C_V\frac{dT}{dt} + P_0\frac{d(1/\rho)}{dt} + \gamma\rho C_V T\frac{d(1/\rho)}{dt}\right]dm \tag{9.21}$$

where the expression for E_G comes from eq 9.18. But the first and third terms in the integral cancel because of the last part of eq 9.20. The last term is small because we can estimate that $d(1/\rho)/dt=(\alpha.dT/dt)/\rho$, where α is the coefficient of thermal expansion, and $\alpha T \ll 1$, but can also be combined with the earlier thermal term using the thermodynamic identity (Maxwell relation) $C_p = C_V(1+\alpha\gamma T)$. So we finally get, to an excellent approximation,

$$L = \int_0^M \left[-C_P\frac{dT}{dt} + Q\right]dm \tag{9.22}$$

(The extent to which this is approximate rather than exact is not straightforward to estimate analytically. Numerical evaluation does however indicate that it is a very good approximation).

What this result means is that as a planet cools and contracts, the gravitational energy becomes more negative and the dominant part of the internal energy (the part due to compression) becomes more positive, but that these cancel! The consequence is that the dominant energy output associated with the planet evolution (aside from radioactivity and external sources of heating) is the change in the thermal energy at constant pressure. This seems intuitively obvious but it is nonetheless widely misunderstood (e.g., there are books which say that Jupiter's energy output is derived from contraction when in fact the dominant effect is simply cooling. Of course, cooling *implies* contraction, but the energy available is nonetheless thermal).

The result is sometimes stated with C_V replacing C_P but this is often a minor point since these differ typically by less than 10%, at least for Jupiter and Saturn (unlike "hot Jupiters" or brown dwarfs). Very importantly, the result does not include gravitational energy release due to compositional changes (e.g. core formation, differentiation in general). This can be immediately recognized by noting that changes in density (determining changes in gravitational energy) do not only come about from work done by pressure... they can also arise from moving constituents around.

The result derived here for luminosity is very different from non-degenerate bodies! As discussed earlier (eq 9.15), a non-degenerate star that loses heat as it contracts is nonetheless getting hotter internally, i.e. it behaves as though it has negative thermal capacity.

We can use the Virial theorem to estimate *very roughly* the low mass limit of main sequence stars. As a star forms and contracts, it has a temperature internally that is given by the Virial theorem: $T \sim GMm_p/k_BR$ (where m_p is the proton mass). It has two possible fates: If this T reaches $\sim 3 \times 10^6$ K before the interior becomes degenerate, then it will ignite thermonuclear reactions. If instead the Fermi pressure becomes more important than the thermal pressure before ignition then the body will stop contracting and settle onto a degenerate cooling curve described by the luminosity equation given above. The Fermi pressure is $\sim 50(M/R^3)^{5/3}$ Megabars (with M and R in units of the mass and current radius of Jupiter) and increases more rapidly than the thermal pressure $\sim 140(M/R^3)(T/10^6\text{K})$ Megabars as R decreases. (This pressure includes the electrons treated as classical particles). In Jupiter units, the Virial $T \sim (0.1 \times 10^6)(M/R)$. This allows for the thermal energy stored in classical electrons. Ignition therefore occurs at $M \sim 30R$. At ignition, this gives $P_{\text{thermal}} \sim 420M/R^3 \sim 12000/R^2$ and $P_{\text{Fermi}} \sim 15000/R^{10/3}$. These are equal at $R \sim 1.3$ and $M \sim 40$. This is only very rough! The correct answer is actually 70 to 80 Jupiter masses. But notice that a Jupiter radius is indeed in the correct ballpark of being a special size: The size at which degeneracy competes with thermal pressure when T is at the lowest temperature for sustained fusion.

Ch. 9 Problems

9.1) In giant planets, $P \propto \rho^2$ cannot be even approximately true in the atmosphere, which is an ideal gas with $P=K_1\rho^{1+\gamma}$ say, where γ is ~0.45 (an adiabatic gas mixture of hydrogen and helium).Consider a Jupiter model in which $P=K\rho^2$ for $\rho>\delta\rho_c$ (where $\delta<<1$ and ρ_c is the central density) and $P=K_1\rho^{1+\gamma}$ for $\delta\rho_c > \rho > 0$, with $\gamma>1$. (Why do we need this constraint on γ?) The pressure is assumed to be continuous at $\delta\rho_c$. Show that to a good approximation, the fractional change in radius of the planet is given by $\delta(1-\gamma)/2\gamma$. [Notice that this gives no change when $\gamma=1$, as it must]. What consequence does this have for the estimated radius of Jupiter relative to the prediction for a polytrope?

Solution: At densities above $\delta\rho_c$ the solution is $\rho_c \sin kr/kr$ with k defined as usual. It immediately follows that the radius R_i at which $\rho(R_i) = \delta\rho_c$ is well approximated by $\pi(1-\delta)/k$. Equating pressures at R_i, we get $K_1 = K(\delta\rho_c)^{1-\gamma}$. Solving hydrostatic equilibrium external to R_i, we can assume $g(r) = GM/r^2$ where M is the constant total mass, because the atmospheric mass is small (i.e., quadratic in the small parameter δ). Let the new outer radius be $\pi(1+\varepsilon)/k$ so $\varepsilon(<<1)$ is the parameter we are seeking to determine. Solving hydrostatic equilibrium in the thin low density outer region, we obtain $(\delta\rho_c)^\gamma = \gamma GMk(\delta+\varepsilon)/(\gamma+1)K_1\pi$. This only works if $\gamma > 1$ since otherwise the density never goes to zero.(To get this result we approximate $1/(1-\delta)$ by $1+\delta$, etc.) Substituting for K_1 and remembering that $M= (3\rho_c/\pi^2).4\pi(\pi/k)^3/3$ and the formula for k in terms of K, we finally get $\varepsilon = \delta(1-\gamma)/2\gamma$. For reasonable choices ($\gamma=0.45$, $\delta=0.03$), this predicts less than a 1% increase in Jupiter's radius relative to the pure n=1 polytrope. It does, however, predict a change that depends on mass since the correct choice of δ is smaller for more massive planets. This means that there should be a mildly increasing radius as one goes to lower masses. Plausibly, a planet with the mass of Saturn but the same composition as Jupiter would be several percent larger radius than Jupiter.

9.2) In a planet such as Jupiter or Saturn (with $P=K\rho^2$) we found that the density solution is eqn 9.6 with $k=(2\pi G/K)^{1/2}$. We chose B=0 because we wanted this solution to apply and be finite at r=0. Under those circumstances, the radius of the planet (R) is given by $kR=\pi$. But suppose the planet possesses a small, very dense core (e.g. of rock or ice). The solution 9.6 still applies *but only outside the core* so we can no longer assume B=0 and we will no longer have $kR=\pi$. Let the core have mass M_c and require (as you must by Gauss' law) that the gravitational acceleration is continuous (i.e. $g = GM_c/R_c^2$ just outside the dense core where R_c is the core radius).

(a) By using hydrostatic equilibrium just outside the core, prove that to a good approximation $B=M_c k^3/4\pi$. ["Good approximation" means that you should make use of the fact that $kR_c<<\pi$. With this assumption, you should be able to prove that the term involving B dominates the density gradient and hence pressure gradient just outside the core. Use series expansions on the sine and cosine functions, i.e. $\sin(x) = x - x^3/6 + ..$].

(b) Find an *approximate* solution for the fractional change in the radius of the planet relative to the case of no core. It should be in the form: fractional change in radius equals something simple involving only A and B. ["Approximate" means of course that you replace $\sin(\pi - x)$ by x if $x \ll 1$, etc].

(c) Hence estimate the mass of Saturn's core (as a fraction of the total mass), assuming that this alone explains the fact that Saturn has smaller radius than Jupiter. (Notice that the result is independent of the core density, provided it is much more dense than hydrogen of course.) If this takes more than three lines, then you're doing it the hard way.

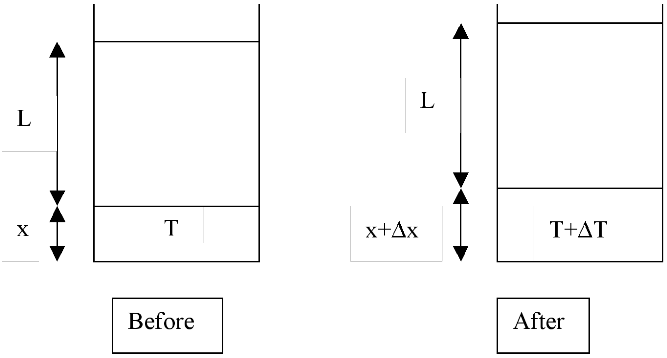
(d) How does this result (part b) compare with the predicted radius for the case where the heavy material in the core is simply mixed throughout the entire volume of the planet (i.e., there is no core)?

9.3) (a) In the derivation of the result for the luminosity of a planet, there are two terms that cancel, dE_o/dt and $P_0 d(1/\rho)/dt$. How large is the magnitude of each of these terms compared to the dominant term that remains ($C_v dT/dt$)? In answering this question, it is useful to introduce a parameter that equals the average or typical value of P_o/K_T. (This parameter is ~2 for Jupiter and ~3 or 4 for deep Earth). If you do this then you need never insert any numerical values into the equations but can answer the question entirely in terms of known dimensionless numbers.

(b) Suppose that the planet has a simple first order phase transition somewhere within. ("Simple" means that it can be treated as univariant, so that there is no change in composition). As the planet cools, there is net change of material from one phase to the other because of the temperature dependence of the phase transition (in accordance with the Clausius –Clapeyron equation). Show that the only consequence for the luminosity is latent heat absorbed or emitted (i.e., there is no consequence from the change in density and resulting change in gravitational energy).

(c) Suppose the core of Mercury froze almost completely in a billion years. How does the energy released as latent heat (and contributing to luminosity) compare with a typical estimate of radioactive heat during the same period? For the latent heat, you should assume $k_B T_m \ln 2$ per iron nucleus where T_m is 2200K. (It should be obvious where this estimate comes from). For radiogenic heat production you can assume 1.5×10^{-7} erg/sec per gram of the rocky component only. Mercury is about 25% rock and 75% iron by mass.

9.4) Consider a vertical column of material confined between rigid walls, as shown. The lowermost portion, thickness x, is heated by an amount ΔT and this causes that layer to grow in thickness by Δx. The material is assumed to be incompressible except for the effect of thermal expansion, and so the entire column above this bottom layer is raised by the same amount. There is a gravitational acceleration g everywhere (and constant) but everything is hydrostatic before and after. Evidently, gravitational work is done, PV work is done and thermal energy has changed (three terms in the energy budget). Show that despite this complexity, the energy that must be supplied to do all this is merely $C_p\Delta T$ per unit mass of the heated region. Do this explicitly; do not use the virial theorem.



Commentary: Notice that this means that you do not have to do explicit gravitational work to build a mountain on Earth, if your method is thermal expansion of the underlying rock material. (This is not widely understood in the earth science community!) By "explicit", I mean that the energy that is needed to make a mountain of a given height has nothing to do with gravity… it would be the same on Earth as on Mars, provided it were done by thermal expansion alone. Make sure you understand this counterintuitive result. (Real mountains are not usually made by thermal expansion alone but the expansion of underlying rock, as in a mantle plume or convective upwelling is nonetheless often important).

9.5) Consider a "puffy" exoplanet with Jupiter mass but radius 1.5 times Jupiter radius. Such a body will not be well described by $P\propto\rho^2$ because the thermal pressure is large (i.e., the body is not strongly degenerate). Nonetheless, we can get a rough idea of the needed internal temperature as follows.

(a) Assuming $P = K\rho^2$, find the value of K required for PJ (Puffy Jupiter) and compute the pressure at it's center. Now compute the pressure inside OJ (Our Jupiter) *at the same density* (not the center obviously!) and attribute the difference to a thermal effect, given by $\gamma\rho C_v\Delta T$, where the Gruneisen $\gamma=0.6$, $C_v=2.5 \times 10^8$ erg/g.K, and ΔT is the temperature difference between PJ and OJ at that density. What is ΔT? For reference, the temperature inside OJ is roughly $6000\rho^{0.6}$ (with ρ in g/cc) but you should expect to find a much larger T inside PJ.

(b) Show that the T you estimate for PJ is achievable from the energy of formation, assuming no thermal energy loss. Hint: Use Viral theorem.

Commentary: The point of this is to show you that a body such as PJ is possible without additional energy sources. In other words, the difficulty with explaining PJ is not necessarily with the First Law (i.e., the need to find an energy source) but rather with avoiding the loss of a lot of internal energy (through integrated luminosity over its history).