1 Introduction: Why study the insides of planets?

Exploration of planetary interiors is difficult. Actually traveling to the inside of planets is orders of magnitude more challenging than traveling between them. Direct observation is similarly problematic, with electromagnetic radiation, including light, easily propagating between planets but scarcely within them. Scientific motivation, however, is not tied to accessibility but to relevance. We should never confuse the importance of investigation with the amount of information available (or the number of investigators). There are several reasons why it is important to study planetary structure and evolution.

First, *it provides the basis for explaining many phenomena that arise within a planet, but manifest themselves externally.* You can't understand why certain planets have volcanoes, why some planets have magnetic fields (and others do not), and why Jupiter emits as much heat as it does without an understanding of the insides of these bodies.

Second, *it provides an essential part of the understanding of external phenomena and processes that are indirectly linked to the interior.* For example, you cannot claim any understanding of the history of Mars' atmosphere if you are ignorant of likely out-gassing rates or of how the sputtering of the atmosphere is affected by a magnetic field that varies through geologic time.

Third, *it provides a unifying framework.* If you want to build a story of how the planet formed and evolved to the present state, this will depend in large part on understanding the interior.

Fourth, *it provides a testing ground for fundamental physics—theory and experiment—under "extreme" conditions.* For example, the most common metal in the universe is probably metallic hydrogen. This material is of fundamental interest to condensed matter physicists. It also happens to dominate the mass of Jupiter. It is not yet well understood, either theoretically or experimentally.

1.1 What is a Planet?

This is not an important question! But it must be posed, if only to explain why it is unimportant. Science deals with things as well as ideas, and we have to give these things names in order to talk to each other with some mutual understanding. It is not wise to spend a lot of time worrying about how we decide on names and categories—it follows that some people are not wise. Since we are interested in processes and evolution, we certainly don't want to settle on a definition as ridiculous as, say, "things that orbit a star and are bigger than X," where X is some arbitrary number. It seems wise to talk about objects that are big enough that there is an interesting consequence arising from their size, usually because of their gravity. Nevertheless, we should honor the usual idea of stars by excluding things so massive that they have, or once had or will have, significant thermonuclear fusion.

Notice that our process-based "definition" says nothing about whether the planet is orbiting a star. Ganymede is clearly a planet. So, for our purposes, is Pluto. Planets may even exist in interstellar space. (Planet means wanderer from the original Greek; the everyday layperson idea of a planet is not at all a wanderer except from a pre-Copernican viewpoint.) We would presumably exclude the myriad small bodies orbiting in the rings of Saturn—they may have different insides from outsides, but that's because of purely surficial processes such as sputtering and collisional erosion. But prescribing a precise range for mass or radius makes no sense because the extent to which a body is affected by its size depends on its make-up. This motivates the following question.

1.2 How does a Planet differ From a Rock or a Cloud of Gas?

At first glance, we might think that the answer lies in a planet's non-uniform composition. Many planets can be thought of as a series of layers of differing materials; perhaps this is what sets planets apart from the materials they are made of. To test this assertion, we could imagine a planet that has the same composition at its center as at its surface. Even in this case, we would find that the planet composed of X behaves very differently from a small chunk of X. This is because the answer to our question lies not in the composition, but in the conditions the material is subjected to. A planet differs from small amounts of the same material for two reasons: (1) Effect of gravity, leading to high pressures and a change in material properties; and (2) Inability of the planet to eliminate heat, and therefore maintaining large internal temperatures. (These temperatures are close to or at the melting point for solid planets, and hotter still for gaseous planets due to adiabatic heating effects.)

1.2.1 Effect of Gravity

The force of gravity plays a primary role in determining the behavior and properties of a planet. Unlike a small amount of matter, which is held together by electronic forces alone, planets are also bound together by self-gravitation. Thus a planet can be thought of as a collection of mass sufficiently large for its matter to be bound together into a single unit, rather than many discrete pieces of matter executing separate orbits, as is the case for Saturn's rings. A planet might be thought of as an object massive enough that the effect of gravity significantly alters the properties of the materials it's made of.

Since gravity is an attractive force, it does work on planet-forming materials by compressing them, and thereby changing their internal energy. Seen another way, anytime matter is made more compact, its gravitational energy decreases—becomes more negative—and therefore its internal energy must also increase to obey conservation of energy. The gravitational energy per particle for a planet is $E_i^{\text{grav}} \sim -GM\mu_i / R$, where G is the gravitational constant, M is the planet's mass, μ_i is the mass of the atom or molecule in question, and R is the planet's radius. If we replace M by $4\pi\rho_{\text{av}}R^3/3$ where ρ_{av} is the average density of the planet, then the gravitational energy is of order $E_i^{\text{grav}} \sim -4\pi G\mu_i\rho_{\text{av}}R^2/3$. We would like to know how much total mass must be present in order to significantly alter the properties of the material making up the planet. For the sake of comparison, we consider typical electronic (or cohesive) energies for tightly bound materials, e.g. rocks or metals, of about ~1eV per atom. Since the electronic energy depends on the atomic separation, this energy will change by an amount of the same order if we significantly change the atomic spacing. Compressing the material will raise its internal energy by forcing the atoms closer together, but lower the gravitational energy by decreasing R at fixed M. It is interesting to determine the critical size of a body, where gravitational and atomic energies are comparably important. This size for rocky planets is given by

$$\frac{4}{3}\pi G\mu\rho_{\text{av}}R^2 \sim 1\text{eV}$$
$$\frac{10}{3}(7\times10^{-8})(30)(1.6\times10^{-24})(3)\left(\frac{R}{1000\text{ km}}\right)^2(10^{16}) \sim 1.6\times10^{-12} \qquad (1.1)$$
$$\Rightarrow \quad R\sim \text{ few thousand km}$$

(All we're seeking here is order of magnitude. So four pi is about ten, G is about 7 x10^{-8} in cgs, mean molecular weight is taken to be thirty times the proton mass, mean density is taken to be that of uncompressed rock (including iron), ~3 g/cc, and an electron volt is 1.6 x 10^{-12} erg.) We should be generous in our assessment and admit rocky bodies at around 1000 km or above, based on the idea that even modest changes in material properties are important. As we shall see below, an even more generous definition is justified by the thermal considerations.

If we consider the same calculation for weaker hydrogen-bonded materials like ice, we require a much smaller energy of only ~0.1 eV, in comparison to the stronger covalently bonded rocks. This order of magnitude decrease in energy dominates over the modest decrease in density for ice relative to rock, reducing the cutoff size to ~1000 km and the cutoff mass even more so. Still weaker materials, such as methane ice, are significantly altered within even smaller bodies.

Another roughly equivalent way to think about this is to ask, "When does the internal pressure become interesting?" meaning that the pressure is large enough to have a significant effect. We can obtain an expression for this threshold pressure using dimensional analysis alone, though we will look at this in much more detail in chapter 3. Pressure has units of force over area, so the pressure required to balance gravity must be of order $P \sim GM^2/R^4 \sim GM^{2/3}\rho_{\text{av}}^{4/3} \sim G\rho_{\text{av}}^2R^2$, see problem (1.1). This dimensional analysis approach is quite useful when it is easy to determine a relevant crossover pressure, as would be the case for giant gas balls. For example, if you thought that a few kilobars (few x 10^9 dynes/cm^2) is an "interesting" pressure for hydrogen—because it is what's needed to squeeze hydrogen to a state that is very different from an ideal gas at room temperature—then for $\rho_{\text{av}} \sim 0.1\,\text{g/cc}$, the requisite mass is ~0.1 Earth masses. This pressure criterion would also lead one to conclude that a protoplasmic planet (a planet made of folded protein chains, say) could be quite small because protein, or living matter in general, is very sensitive to pressure changes.

1.2.2 Effect of Internal Heating

Somewhat smaller numbers for a critical size arise when we consider whether the inside of the body is hot. In order to get a rough idea of the thermally determined threshold size, we can simply scale measured values for the Earth. The Earth's near surface conductive temperature gradient is roughly -15 K/km. (This was already well known by deep diamond miners in South Africa in the nineteenth century and figured prominently in Lord Kelvin's ideas about conditions inside Earth.) Fourier's law of heat conduction, covered in more detail in chapter 8, states that the heat flow (units of energy per unit time) due to conduction is proportional to cross-sectional area and the temperature gradient across that area. Irrespective of whether this heat flow is arising from radiogenic heat production that escapes, or steady cooling of the interior, or a combination of the two (they are comparably important for Earth), this heat flow will scale approximately as the volume $H \propto R^3$. Using the fact that the surface area goes as $A \propto R^2$, we can rearrange the law of conduction to find that the conductive thermal gradient is proportional to the planet's radius

$$\frac{dT}{dr} \propto \frac{H}{A} \propto \frac{R^3}{R^2} \propto R \tag{1.2}$$

Now that we have used our physical understanding to find how a planet's temperature gradient depends on its radius, we can create a scaling relation by dividing the equation above with the same measured values for the Earth. By doing this, we are assuming that the constants of proportionality that we neglected to work out in detail are roughly the same for the Earth and the new planet we are comparing to. We finally rearrange to yield our scaling relation.

$$\frac{dT}{dr} \sim \left(\frac{-15\text{ K}}{\text{km}}\right)\left(\frac{R}{6400\text{ km}}\right) \tag{1.3}$$

Notice that if we plug back in values for Earth, we obtain the Earth's thermal gradient just as expected. This technique of determining how a process depends on different variables and then creating ratios using measured values from a known example is what we mean by a *scaling argument*. We will make frequent use of scaling arguments in this book.

In the limit of a small body, where we might not expect convection to matter, the peak internal temperature would then be raised above the surface temperature by an amount $\Delta T \sim (dT/dr)\,\Delta r$, or using our scaling relation

$$\Delta T \sim \left(\frac{15\text{ K}}{\text{km}}\right)\left(\frac{R}{6400\text{ km}}\right)R \tag{1.4}$$

Similar to the case of determining a critical size for compression due to gravity, we must now decide on what a "significant" temperature change is. For rocky bodies, an increase of roughly 1000 K can easily be considered significant, as this brings rocks to their melting point at ambient pressure. Using equation (1.4), we can see that the temperature change is a few 1000 K for $R \sim 1000\text{ km}$. For an icy body, you might only need a temperature rise ~100 K to reach melting, which would be achievable for a body that is only $R \sim 300\text{ km}$. As mentioned briefly above, this assumes similar thermal conductivities for ice and rock and similar heat production, which both happen to be true (though this argument does not work for tidally heated bodies).

1.2.3 Comparison of Planets with Stars

In our attempts to understand planetary interiors, it is useful to examine the similarities and differences between stars and planets. Unlike planets, stars have a life cycle that strictly governs their behavior. Their early evolution (while on the main sequence), involves thermonuclear fusion of hydrogen. The rate of hydrogen burning depends strongly on mass, and thus very massive stars exhaust their hydrogen fuel quickly while the lowest mass stars evolve so slowly that they are still stuck in this stage after entire age of the universe. Once a star exhausts its fuel and begins to evolve away from the main sequence, it has a variety of possible end points depending on its mass. For more massive bodies, white dwarfs and neutron stars are possible end points; these bodies have more in common with planets than main sequence stars, because they derive their pressure support from the condensed matter properties of their constituent materials rather than from thermal pressure.

But for the moment, let's focus on main sequence stars like our Sun. Planets differ fundamentally from these stars because their internal energy is not primarily thermal energy. Suppose we assume that all of the gravitational energy released in building up a massive body was retained and stored as thermal energy. This is essentially the same as saying that the body has not significantly cooled since formation, which is a reasonable assumption for main sequence stars. The predicted temperature of formation would then be given by

$$\frac{GM\mu}{R} \sim (\text{few})\, k_B T \tag{1.5}$$

where k_B is Boltzmann's constant and the "few" comes from the degrees of freedom available for the storage of heat (we will do this more precisely later on). By replacing the total mass with density times volume, the following scaling relation can be obtained

$$T \sim 10\left(\frac{\rho_{\text{av}}}{1\ \text{g/cm}^3}\right)\left(\frac{R}{1000\ \text{km}}\right)^2\left(\frac{\mu}{m_{\text{p}}}\right) \tag{1.6}$$

where m_p is the proton mass. This predicts $T \sim 10^4\,\text{K}$ for a Mars sized body and a few times larger for an Earth sized one. The actual internal temperatures of planets, however, are significantly smaller—for example the Earth is cooler by about a factor of ten—because planets can cool efficiently. In contrast, main sequence stars are maintained at their temperatures of formation by thermonuclear reactions.

Another way of stating this is in terms of the thermal expansion state of the planet or star. The coefficient of thermal expansion is a material property that is equal to the expected fractional change in volume per degree change in temperature (at constant pressure), $\alpha = (dV/dT)_P / V \approx (\Delta V/V)/\Delta T$. Thus we can estimate the total change in volume due to the thermal state of the body by multiplying by temperature

$$\left(\frac{\Delta V}{V}\right)_{\text{thermal}} \sim \left(\frac{\Delta V/V}{\Delta T}\right)T \sim \alpha T \tag{1.7}$$

As we will do quite often in this book, we consider a fractional change of one to be highly significant, and therefore we are interested in the value of αT relative to 1. An important special case of the thermal expansion state, $\alpha T = 1$, occurs for bodies that are composed entirely of an ideal gas—as is roughly the case for stars—because volume is linearly proportional to temperature at constant pressure—indeed αT is close to unity for the Sun. Planets, on the other hand, are quite different with much smaller values of $\alpha T < 0.1$. This means that the size of planets is fairly insensitive to their internal temperature; another way of saying this is that *planets are degenerate.* This has a precise double meaning in the language of condensed matter physics: It means that the vast majority of the electrons are in their ground state, but it also means that the compressibility properties of the material are predominantly those of condensed matter rather than those of a gas. A gas giant, such as Jupiter, is on average just about as degenerate as a terrestrial planet like Earth, even though its outer region (a tiny fraction of its mass) is indeed "gas" in the everyday sense of the word. "Hot" Jupiters, many of which have been recently discovered, are not as close to the degenerate limit (although most of their mass is still degenerate). As a consequence, thermal expansion is not negligible and they can have significantly larger radius than Jupiter

Degeneracy is so important to planets that it is part of their definition: Hydrogen-rich bodies less massive than about 0.08 solar masses never achieve hydrogen burning and cool to a degenerate state, resulting in a brown dwarf or giant planet. Planets may be "cold" in the sense that their thermal energies are small compared to electronic energies, but they are not cold in the sense of being well below the melting point or Debye temperature of relevant materials.

Even if they are unfamiliar now, all of these things will become clearer as we proceed—*DON'T PANIC.*

Ch. 1 Problems

1.1) It says in the text that "the pressure required to balance gravity must be of order $P \sim GM^2/R^4 \sim GM^{2/3}\rho_{\text{av}}^{4/3} \sim G\rho_{\text{av}}^2 R^2$." Confirm that this is indeed an estimate for the central pressure of a planet. Do this by dimensional analysis alone—do not use hydrostatic equilibrium! Estimate the radius of the largest pure water ice body that has a central pressure less than the pressure at which Ice I (everyday ice) converts to Ice II, i.e., 2 kilobars.

1.2) It has been suggested that biological processes do not operate well at pressures above a few kilobars. (It's probably not true, but that's a whole other story not told here.) Assuming that a "living" planet, i.e., a planet made entirely of living matter with a density of ~1 g/cc, must have an internal pressure that is less than this value, what is the largest living planet?

Solution: Using $P \sim G\rho_{\text{av}}^2 R^2 \sim 7\times10^{-8}(1)(10^{16})(R/1000\text{ km})^2$, and P=2 or 3 x 10^9 dynes/cm^2, we get R~2000 km.

1.3) The abundance of radioactive elements within a planet depends on the environment in which the planet formed. Plausibly there are places in the Universe where the local abundance is a factor of ten higher than in our solar system. How would this affect the "lower bound" on planet size discussed above?

1.4) Uranium-238 is responsible for a large fraction of heat leaving Earth and is present with an abundance of ~ 2 x 10^{-8} by mass. It has a half-life that is ~ 10^{10}

longer than for Polonium-210, but the energy released per nuclear decay is similar. If we define a "planet" as a body that has a large temperature difference between inside and outside, then what is the smallest planet of pure ^{210}Po? Could it be concealed in a glass of vodka (or saki)? Note: You are only expected to do this to order of magnitude. This means you should not waste time worrying over such things as how different the density of Po is relative to rock (it is ~ 20 g/cc). What matters is the huge order of magnitude difference that comes from applying the scaling described in the text when you changes the decay rate and abundance by so many orders of magnitude. You should, of course, expect to get a "ridiculous" answer! Can you see why this answer might actually be wrong (as well as ridiculous)? That is, what additional physics might you need to consider?

[^{210}Po was used in a murder in late 2006 in London. See, for example, *http://www.physicsforums.com/showthread.php?t=145253*]